SPHERICAL FUNCTIONS ON SPHERICAL VARIETIES

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Abstract. Let $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ be a spherical variety for a split reductive group $\mathbf{G}$ over the integers $\mathfrak{o}$ of a p-adic field k, and $K = \mathbf{G}(\mathfrak{o})$ a hyperspecial maximal compact subgroup of $G = \mathbf{G}(k)$. We compute eigenfunctions ("spherical functions") on $X = \mathbf{X}(k)$ under the action of the unramified (or spherical) Hecke algebra of G, generalizing many classical results of "Casselman-Shalika" type. Under some additional assumptions on $\mathbf{X}$ we also prove a variant of the formula which involves a certain L-value, and we present several applications such as: (1) a statement on "good test vectors" (namely, that an H-invariant functional on an irreducible unramified representation π is always non-zero on π^K), (2) the unramified Plancherel formula for X, including a formula for the "Tamagawa measure" of $\mathbf{X}(\mathfrak{o})$, and (3) a computation of the most continuous part of $\mathbf{H}$-period integrals of principal Eisenstein series.

Contents

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1. Introduction

1.1. **The problem.** Let $\mathbf{G}$ be a split reductive group over the ring of integers $\mathfrak{o}$ of a local non-archimedean field k in characteristic zero (with residual degree q, a power of the prime number p) and let $\mathbf{X}$ be a homogeneous spherical scheme for $\mathbf{G}$ over $\mathfrak{o}$ (i.e. a $\mathbf{G}$-scheme on which the Borel subgroup $\mathbf{B}$ has an open orbit – this includes, but is not limited to, symmetric spaces). Then $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$, where $\mathbf{H}$ is the stabilizer of an $\mathfrak{o}$-point x_0 in the open Borel orbit ($\mathbf{HB}$ is open in $\mathbf{G}$). We make throughout certain assumptions on $\mathbf{X}$ (cf. §1.7), including that $\mathbf{X}$ is quasi-affine, which cause no serious harm to generality. We denote by regular font the corresponding sets of k-points of each of the groups. We also denote $K = \mathbf{G}(\mathfrak{o})$ – it is a (hyperspecial) maximal compact subgroup. To avoid complicated formulas, we will assume that k is unramified over the field $\mathbb{Q}_p$; the modifications needed to remove this assumption are trivial.

We consider the right regular representation of G on $V = C^\infty(X)$ (if G has a unique orbit on X, this is just the induced representation $\mathrm{Ind}_H^G(1)$). The unramified[1] Hecke algebra $\mathcal{H}(G,K)$ acts on V, and the goal of this article is to compute an explicit formula for its eigenvectors, expressed in terms of the geometry of $\mathbf{X}$. More generally, if $\mathbf{H}$ contains the unipotent radical $\mathbf{U}_P$ of a proper parabolic, $\Lambda : \mathbf{U}_P \to \mathbb{G}_a$ is a homomorphism which is fixed under $\mathbf{H}$-conjugation and $\psi : k \to \mathbb{C}^\times$ is a character, then we can also consider the space $V = C^\infty(X, \mathcal{L}_\Psi)$, where $\Psi = \psi \circ \Lambda$ considered as a character of H and $\mathcal{L}_\Psi$ is the complex line bundle defined by it. (This space is just $\mathrm{Ind}_H^G(\Psi)$ if G has unique orbit on X.) For an explanation of how this case can be understood geometrically, cf. [Sa08, §5.5]. Though most of the results carry over verbatim to this "twisted" case, for simplicity we do not discuss it in the introduction.

This is a problem of both harmonic-analytic and arithmetic interest, and there is a long history of particular examples which have been computed [Ma71, Ca80, CS80, HS88, HS00, Hi99, Hi05, KMS03, Of04, Sa06, to mention just a few]. In the first part of this paper, we compute a very general formula, covering all previously established cases (when the group $\mathbf{G}$ is split) and many more. Besides its case-specific arithmetic applications, which have been the motivation for most of the literature, the computation of such a formula is relevant to the $\mathbf{H}$-period integrals of principal Eisenstein series (which we undertake in §10), and conjecturally also to the $\mathbf{H}$-period integrals of other automorphic forms, via the local Plancherel formula (which we develop in §9). These applications, presented in the third part of the paper, require an improved version of the general formula, which we develop in the second part and which, in particular, involves a certain local L-value of the unramified representation in question.

Before we proceed to a more detailed description of the results, let us put this work in a more general context, under the perspective of automorphic forms. The study of period integrals of automorphic forms (a major stream of which is related to the relative trace formula of Jacquet – see [Lap06] for a presentation) has revealed relationships between the non-vanishing of certain period integrals of automorphic forms (typically, over spherical subgroups) and functorial lifts, on one hand, and between the values of these period integrals and L-functions or special values thereof,

[1] In the literature, the term "spherical" is often used for "unramified", but to avoid confusion I will reserve this term for the notion of spherical varieties.

on the other. Though no general theory or conjectures exist to describe these phenomena, it has recently started to become clear that a general and systematic approach should be possible. The "dual group" attached by Gaitsgory and Nadler [GN], in the context of the Geometric Langlands program, to any spherical variety $\mathbf{X}$ was shown in [Sa08] to be related to unramified representations in the spectrum of X and is, more generally, conjectured in [SV] to be describing X-distinguished representations, in a certain sense. Moreover, under certain assumptions period integrals are (roughly speaking) conjectured to be "Eulerian" with local Euler factors equal to "the H-invariant functionals which appear in the local Plancherel formula for X". The last step is to relate these H-invariant functionals to special values of L-functions, and this is part of what we accomplish here, when the local group is split and the representation is unramified.

1.2. **The general formula.** To formulate the general formula, we will for simplicity assume in the introduction that $B = \mathbf{B}(k)$ has a *unique* open orbit on X. This includes, in particular, the *multiplicity-free* case, i.e. the case when the eigenspaces for the unramified Hecke algebra are one-dimensional.[2] Then orbits of K on X are in bijection $x_{\check\lambda} \leftrightarrow \check\lambda$ with anti-dominant, in a suitable sense, cocharacters into $\mathbf{A}_X$, where $\mathbf{A}_X$ is a quotient of a maximal torus $\mathbf{A}$ of $\mathbf{G}$ identified with the $\mathbf{A}$-orbit of a chosen point $x_0 \in \mathbf{X}(\mathfrak{o})$.

Let A^* denote the "Langlands dual" complex torus of $\mathbf{A}$ (the torus of unramified complex characters of A) and let A_X^* denote the "Langlands dual" of $\mathbf{A}_X$. Naturally (under the assumptions of the introduction) $A_X^* \subset A^*$. There is a surjective map $A^* \to \operatorname{spec}_M \mathcal{H}(G, K)$ with generic fiber finite of order $|W|$, where W is the Weyl group of G. The support of V^K as a module for $\mathcal{H}(G,K)$ coincides with the image of a translate $\delta_{(X)}^{\frac12} A_X^*$ of A_X^*, see [Sa08]. The torus A_X^* (and its translate $\delta_{(X)}^{\frac12} A_X^*$) carries a natural action of a finite reflection group $W_X \subset W$, called the "little Weyl group" of X, and the map $\delta_{(X)}^{\frac12} A_X^* \to \operatorname{spec}_M \mathcal{H}(G,K)$ factors through the topological quotient $\delta_{(X)}^{\frac12} A_X^*/W_X$. We can now formulate our general formula:

1.2.1. **Theorem** (cf. Theorems 4.2.2, 5.2.1, see also (6.2))**.** *For almost all points $\eta \in \operatorname{spec}_M \mathcal{H}(G,K)$ in the support of V^K, the corresponding eigenspace in V^K admits a basis consisting of the functions:*

$$\Omega_{\delta_{(X)}^{\frac12}\chi}(x_{\check\lambda}) = e^{-\check\lambda}(\delta_{P(X)}^{\frac12}) \sum_{w \in W_X} B_w(\chi) e^{\check\lambda}({}^w\chi) \tag{1.1}$$

where: $\delta_{(X)}^{\frac12}\chi \in \delta_{(X)}^{\frac12} A_X^*$ ranges over a set of representatives for the elements of $\delta_{(X)}^{\frac12} A_X^*/W_X$ which map to $\eta \in \operatorname{spec}_M \mathcal{H}(G,K)$ and the coefficients $B_w(\chi)$ are certain cocycles: $W_X \to \mathbb{C}(A_X^*)$ for the computation of which we give an explicit algorithm in terms of the geometry of $\mathbf{X}$. For the definition of $\delta_{(X)}, \delta_{P(X)}$ cf. §2.1.

The cocycles B_w have the following conceptual meaning: It was proven in [Sa08] that certain natural morphisms: $S_\chi : C_c^\infty(X) \to I(\chi)$ satisfy proportionality relations: $T_w \circ S_\chi \sim S_{{}^w\chi}$ for all $w \in W_X$, where $I(\chi) = \operatorname{Ind}_B^G(\chi\delta^{\frac12})$ is an unramified principal series representation and T_w are intertwining operators between principal series. Then the B_w are a standard term times the coefficients of proportionality,

[2] Part of what we show in the third part – under some additional assumptions – is that this is equivalent to the weaker condition that *generic* eigenspaces be one-dimensional.

when T_w are normalized in a "good" way. This "good" normalization comes from the theory of equivariant Fourier transform on the basic affine space $U\backslash G$ [BK98], which we review in §3.8. Use of the equivariant Fourier transform allows us to avoid complicated formulas and mysterious cancellations which are abundant in the relevant literature.

The method employed here is based on the basic idea of Casselman and Shalika [Ca80, CS80], namely that instead of computing an unramified eigenfunction one should first compute an Iwahori-invariant function coming from a vector of small support in the principal series, and then use the fact that the space of Iwahori-invariant vectors has dimension equal to the number of intertwining operators between principal series. The representation-theoretic results of [Sa08] allow us to reduce the computation to the case of $\mathbf{SL}_2$, and there is only a small number of possibilities to consider there.

1.3. **The formula with L-values.** Because of its inductive, algorithmic definition, the general formula may be difficult to use in applications. Therefore, in the second part of the paper we transform it (under additional assumptions, which are in particular satisfied by many *affine* homogeneous spherical varieties) to a more useful one, expressed in terms of the "dual group" $\check{G}_X$ of $\mathbf{X}$ – more precisely the root datum of this group (cf. §6.1). This should be, conjecturally, isogenous to the subgroup $\check{G}$ attached to X by Gaitsgory and Nadler [GN].

1.3.1. **Theorem** (cf. Theorem 7.2.1)**.** *(Under additional assumptions.) We have:*

$$\frac{\Omega_{\delta_{(X)}^{\frac{1}{2}}\chi}(x_{\check{\lambda}})}{\beta(\tilde{\chi})} = \delta_{P(X)}^{-\frac{1}{2}}(x_{\check{\lambda}}) \prod_{\Theta^+} (1 - \sigma_{\check{\theta}} q^{-r_{\check{\theta}}} T_{\check{\theta}}) s_{\check{\lambda}}(\chi) \tag{1.2}$$

where $s_{\check{\lambda}} = \frac{\sum_{W_X} (-1)^w e^{\check{\rho} - w\check{\rho} + w\check{\lambda}}}{\prod_{\check{\gamma}>0}(1-e^{\check{\gamma}})}$ *is the Schur polynomial* indexed by lowest weight *(that is, if $\check{\lambda}$ is anti-dominant then $s_{\check{\lambda}}$ is the character of the irreducible representation of $\check{G}_X$ with lowest weight $\check{\lambda}$) and $T_{\check{\theta}}$ denotes the formal operator: $T_{\check{\theta}} s_{\check{\lambda}} = s_{\check{\theta}+\check{\lambda}}$.*

Here

$$\beta(\tilde{\chi}) := \frac{\prod_{\check{\gamma}\in\check{\Phi}_X^+}(1-e^{\check{\gamma}})}{\prod_{(\check{\theta},\sigma_{\check{\theta}},r_{\check{\theta}})\in\Theta^+}(1-\sigma_{\check{\theta}} q^{-r_{\check{\theta}}} e^{\check{\theta}})}(\chi), \tag{1.3}$$

and the triples $(\check{\theta}, \sigma_{\check{\theta}}, r_{\check{\theta}}) \in \Theta^+$ consisting of a weight of A_X^*, a sign and a positive half-integer are given explicitly in terms of the geometry of $\mathbf{X}$.

If $\mathbf{X}$ is affine then, in particular:

$$\Omega_{\delta_{(X)}^{\frac{1}{2}}\chi}(x_0) = c\beta(\chi) \tag{1.4}$$

with the constant:

$$c = \beta(\delta_{P(X)}^{\frac{1}{2}})^{-1}. \tag{1.5}$$

Notice that the value of $\Omega_{\delta_{(X)}^{\frac{1}{2}}\chi}$ at x_0 is equal to "half" a local L-value for $\check{G}_X$, more precisely $L_X := c^2\beta(\chi)\beta(\chi^{-1})$ is a quotient of L-values for $\check{G}_X$. As we will explain below, this L-value is related to the Plancherel measure for the unramified spectrum of X, and should (conjecturally) be related to $\mathbf{H}$-period integrals of automorphic forms. It would certainly be desirable to have a more natural, geometric

understanding of this L-value, rather than the combinatorial one which we provide here.

The weights $\check\theta$ which appear in this formula are related to the *colors* of the spherical variety, in other words the irreducible $\mathbf{B}$-invariant divisors on $\mathbf{X}$. Each of them induces a $\mathbf{B}$-invariant valuation on $k(\mathbf{X})$ which, by restriction to $\mathbf{B}$-eigenfunctions, defines a weight of A_X^*. Roughly speaking, in the affine case the weights $\check\theta$ contained in the above formula are those W_X-translates of the weights defined by colors which are contained in the positive span of colors.

1.4. **Representation-theoretic results.** A benefit of the last formula is that it allows us to draw certain representation-theoretic conclusions, in combination with the results of [Sa08]. More precisely:

1.4.1. **Theorem** (Multiplicity-freeness, "good test vectors", cf. §8.). *(Under the assumptions of the Theorem 1.3.1, with $\mathbf{X}$ affine.) Assume that for a generic point in the support of V^K the corresponding eigenspace is one-dimensional. Then V^K is a principal module over $\mathcal{H}(G,K)$, generated by the characteristic function of $\mathbf{X}(\mathfrak{o})$. In particular, if π is an irreducible, unramified representation of G and L is a non-zero H-invariant functional on π, then L is nonzero on π^K.*

The last statement in the theorem gives a conceptual answer to non-vanishing questions related to Eulerian period integrals of automorphic forms; namely, it proves that if $\pi = \otimes'_v \pi_v$ is an irreducible representation of the points of a group $\mathbf{G}$ over the adele ring $\mathbb{A}_F$ of a global field F, and $\mathbf{H}$ is a spherical subgroup over F such that $(\mathbf{H}\backslash\mathbf{G})_{F_v}$ satisfies the assumptions of the above theorem for almost every place v, then $\mathrm{Hom}_{H_v}(\pi_v, 1) \neq 0$ for every place v implies $\mathrm{Hom}_{\mathbf{H}(\mathbb{A}_F)}(\pi, 1) \neq 0$.

1.5. **Unramified Plancherel formula.** Another application which we present is the Plancherel formula for the unramified part of the spectrum of X, i.e. for $L^2(X)^K$. More precisely, keeping the assumptions of Theorem 1.3.1 with $\mathbf{X}$ affine, we normalize the invariant measure on X such that $\mathrm{Vol}(x_0 J) = 1$, where J denotes the Iwahori subgroup of G. We denote by $A_X^{*,1}$ the maximal compact subgroup of A_X^* (the subgroup of unitary characters.) Then the Plancherel formula restricted to $C_c^\infty(X)^K$ reads:

1.5.1. **Theorem** (cf. Theorem 9.0.1). *For every $\Phi \in C_c^\infty(X)^K$ we have:*

$$\|\Phi\|^2 = \frac{1}{Q\cdot|W_X|}\int_{A_X^{*,1}} \left|\left\langle \Phi, \Omega_{\delta_{(X)}^{\frac12}\chi}\right\rangle\right|^2 d\chi. \tag{1.6}$$

where $d\chi$ is probability Haar measure on $A_X^{*,1}$ and

$$Q = \frac{\mathrm{Vol}(K)}{\mathrm{Vol}(Jw_l J)} = \prod_{\check\alpha\in\check\Phi^+} \frac{1-q^{-1-\langle\check\alpha,\rho\rangle}}{1-q^{-\langle\check\alpha,\rho\rangle}}.$$

Equivalently, if the eigenfunctions $\Omega_{\delta_{(X)}^{\frac12}\chi}$ were re-normalized to have value 1 at x_0, then the corresponding Plancherel measure on $\delta_{(X)}^{\frac12} A_X^{*,1}/W_X$ would be given by $Q^{-1}L_X(\chi)d\chi$.

This theorem also leads to a computation of $\mathrm{Vol}(\mathbf{X}(\mathfrak{o}))$ (Theorem 9.0.3). This volume is essentially (up to a factor $(1-q^{-1})^{\mathrm{rk}(A_X^*)}$) the local "Tamagawa" volume, by which we mean the volume with respect to an *integral, residually non-vanishing* invariant volume form.

1.6. **Periods of automorphic forms.** Finally, as a direct application of our formula we compute in §10 the "most continuous part" of H-period integrals of principal Eisenstein series, when $\mathbf{H}\backslash\mathbf{G}$ is a spherical subgroup of a split group defined over a global field F (and locally satisfying the assumptions of Theorem 1.3.1) and show that it is given, essentially, by the "half L-value" $L_X^{\frac{1}{2}} := c\beta(\chi)$ of Theorem 1.3.1. More precisely, we consider a pseudo-Eisenstein series on $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)$, which can be analyzed as an integral of Eisenstein series $E(f_\chi, g)$, and show (for simplicity: if the Eisenstein series are induced from characters trivial on the maximal compact subgroup of $\mathbf{B}(\mathbb{A}_F)$):

1.6.1. **Theorem** (cf. Theorem 10.0.2)**.** *The period integral of:*

$$\sum_{\gamma\in\mathbf{B}(F)\backslash\mathbf{G}(F)} \Phi(\gamma g) = \int_{\exp(\kappa+i\mathfrak{a}^*_{\mathbb{R}})} E(f_\chi, g) d\chi \tag{1.7}$$

over $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A}_F)$ *is equal to:*

$$\int_{\exp(\kappa+i\mathfrak{a}^*_{X,\mathbb{R}})} \left(L_X^{\frac{1}{2},S}(\chi)\right)^* \sum_{[W/W_{(X)}]} \left(\tilde{j}_w^S(\chi)\prod_{v\in S}\Delta^{Y,\mathrm{Tam}}_{{}^w\chi,v}(f_{{}^w\chi,v})\right) d\chi \tag{1.8}$$

plus terms which depend on the restriction of f_χ, as a function of χ, to a subvariety of smaller dimension.

For the notation, see §10. The same L-values should show up in the H-period integral of cusp forms, according to an idea of A. Venkatesh which we formulate as an (almost) precise conjecture in [SV].

1.7. **Assumptions and notation.** All our schemes, subgroups etc. are over $\mathfrak{o}$ unless otherwise specified. The group $\mathbf{G}$ is assumed reductive, connected and split over $\mathfrak{o}$. For simplicity, we also assume that the derived group of $\mathbf{G}$ is simply connected. We fix throughout a Borel subgroup $\mathbf{B}$ with unipotent radical $\mathbf{U}$ and a maximal split torus $\mathbf{A}\subset\mathbf{B}$. The choice of $\mathbf{A}$ will be explained in §2.1. The group $\mathbf{A}(\mathfrak{o})$ will also be denoted by A_0, and similarly $B_0 = \mathbf{B}(\mathfrak{o})$, $U_0 = \mathbf{U}(\mathfrak{o})$. The complex torus of unramified characters of A is denoted by A^*. The Weyl group is denoted by W, roots are generally denoted by small Greek letters α,β etc. and the corresponding co-roots by $\check\alpha,\check\beta$ etc. Similarly, ρ denotes half the sum of positive roots and $\check\rho$ denotes half the sum of positive co-roots. We will be using exponential notation for characters of tori, i.e. e^α. The real part $\Re(\chi)$ of a character $\chi = e^\theta$ of a torus is, by definition, the real part of θ. We denote by Φ, Φ^+ and Δ the sets of roots, positive roots, and simple positive roots, respectively (for our choice of Borel). The parabolic corresponding to a set of simple positive roots $\{\alpha,\beta,\dots\}$ will be denoted by $\mathbf{P}_{\alpha\beta\dots}$. We denote by $\mathbb{G}_m$, $\mathbb{G}_a$ the multiplicative, resp. additive group and by $\mathbb{A}^n$ the n-dimensional affine space, sometimes with an index denoting chosen coordinates for this space. For a subgroup $\mathbf{M}$ we will denote by $\mathcal{Z}(\mathbf{M})$ its center, by $\mathcal{N}(\mathbf{M})$ (or $\mathcal{N}_{\mathbf{G}}(\mathbf{M})$) its normalizer, by $\mathcal{R}(\mathbf{M})$ its radical, by $\mathbf{U}_M$ its unipotent radical and by $\mathfrak{d}_M$ its modular character (the quotient of a right- by a left-invariant volume form), considered as a character of $\mathcal{N}(\mathbf{M})$. We denote by δ_M the absolute value of $\mathfrak{d}_M$, and δ_B will simply be denoted by δ. We denote $K = \mathbf{G}(\mathfrak{o})$, and $J =$ the standard Iwahori subgroup (the inverse image of $\mathbf{B}(\mathbb{F}_q)$ under the reduction map $K\to\mathbf{G}(\mathbb{F}_q)$).

We are given a homogeneous spherical scheme $\mathbf{X}$ over $\mathfrak{o}$ such that the action map $\mathbf{G} \times \mathbf{X} \to \mathbf{X}$ is a smooth morphism of $\mathfrak{o}$-schemes. We denote by $\mathring{\mathbf{X}}$ the open Borel orbit. The quotient of a scheme $\mathbf{V}$ (resp. a topological space V) by the action of a group scheme $\mathbf{\Gamma}$ (resp. a topological group Γ), whenever it exists (in the sense of geometric quotient for schemes, or topological quotient in the topological setting), will be denoted by $\mathbf{V}/\mathbf{\Gamma}$ (resp. V/Γ). Thus, the reader should not be confused by expressions of the form G/Γ, where G is a group and Γ is another group acting on it, though not a subgroup of G. We fix throughout a complex character ψ of the additive group k, whose conductor is equal to the ring of integers $\mathfrak{o}$; this character will be used for our Fourier transforms, but also when we consider line bundles $\mathcal{L}_\Psi$ as described in §1.1.

As mentioned above, $\mathbf{X}$ is assumed to be quasi-affine, and such that X admits a G-eigenmeasure and $\mathring{X}$ admits a B-invariant measure. This can always be achieved by a trivial modification, see [Sa08, §3.8]. We also assume that $\mathbf{X}$ satisfies Axioms 2.4.1 and 2.4.2, which is always the case at almost every place if $\mathbf{X}$ and $\mathbf{G}$ are defined over a global field (with $\mathbf{G}$ split).

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Part 1. The sum formula

2. Invariants and orbits

2.1. **Invariants associated to spherical varieties.** We start with the description of certain invariants associated to a spherical variety. For a variety $\mathbf{Y}$ with a $\mathbf{B}$-action, we denote by $k(\mathbf{Y})^{(\mathbf{B})}$ the group of non-zero rational $\mathbf{B}$-eigenfunctions on $\mathbf{Y}$ and by $\mathcal{X}(\mathbf{Y})$ the group of $\mathbf{B}$-eigencharacters on $k(\mathbf{Y})^{(\mathbf{B})}$. If $\mathbf{Y}$ has a dense $\mathbf{B}$-orbit, then we have a short exact sequence: $0 \to k^\times \to k(\mathbf{Y})^{(\mathbf{B})} \to \mathcal{X}(\mathbf{Y}) \to 0$. The *rank* of a $\mathbf{B}$-variety $\mathbf{Y}$ is, by definition, the rank of $\mathcal{X}(\mathbf{Y})$.

Given a spherical variety $\mathbf{X}$ with open Borel orbit $\mathring{\mathbf{X}}$, the *associated parabolic* is the standard parabolic $\mathbf{P}(\mathbf{X}) := \{p \in \mathbf{G} | \mathring{\mathbf{X}} \cdot p = \mathring{\mathbf{X}}\}$. We will denote by $\Delta(\mathbf{X})$ the set of simple roots to which it corresponds. We make once and for all a choice of a point $x_0 \in \mathring{\mathbf{X}}(\mathfrak{o})$ and let $\mathbf{H}$ denote its stabilizer; hence $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ and $\mathbf{HB}$ is open in $\mathbf{G}$. Pick $f \in k[\mathbf{X}]$ such that the set-theoretic zero locus of f is $\mathbf{X} \smallsetminus \mathring{\mathbf{X}}$. Its differential df at $1 \in \mathbf{G}$ defines an element in the coadjoint representation of $\mathbf{G}$, and the centralizer $\mathbf{L}(\mathbf{X})$ of df is a Levi subgroup of $\mathbf{P}(\mathbf{X})$. The absolute value of the modular character of $\mathbf{B}\cap\mathbf{L}(\mathbf{X})$ will be denoted by $\delta_{(X)}$. We fix a maximal torus $\mathbf{A} \subset \mathbf{B}$ with $\mathbf{A} \subset \mathbf{L}(\mathbf{X})$ and define $\mathbf{A}_X$ to be the torus: $\mathbf{A}_X := \mathbf{L}(\mathbf{X})/(\mathbf{L}(\mathbf{X}) \cap \mathbf{H})$. We may consider $\mathbf{A}_X$ as a subset of $\mathbf{X}$ by identifying it with the orbit of $\mathbf{L}(\mathbf{X})$ (or $\mathbf{A}$) through x_0. Let Λ_X be the coweight lattice of $\mathbf{A}_X$; it can be naturally identified with $\mathrm{Hom}(\mathcal{X}(\mathbf{X}), \mathbb{Z})$. Let Λ_X^+ denote the monoid of $\mathbf{G}$-invariant ($\mathbb{Z}$-valued, trivial on k) valuations on $k(\mathbf{X})$; it can be considered as a submonoid of Λ_X by restriction to $k(\mathbf{X})^{(\mathbf{B})}$. (Indeed, no non-trivial $\mathbf{G}$-invariant valuation vanishes on $k(\mathbf{X})^{(\mathbf{B})}$, cf. [Kn91, Corollary 1.8].) Let $\mathcal{Q} = \mathrm{Hom}(\mathcal{X}(\mathbf{X}), \mathbb{Q})$, and let $\mathcal{V} \subset Q$ be the cone spanned by Λ_X^+ in $\mathcal{Q}$. It is known that $\mathcal{V}$ contains the image of the *negative* Weyl chamber, and in many cases (notably, symmetric varieties) coincides with it [Kn91,

§5]. We have a natural bijection: $\Lambda_X \simeq \mathbf{A}_X(k)/\mathbf{A}_X(\mathfrak{o})$, induced from $\check\lambda \mapsto e^{\check\lambda}(\varpi)$. Elements of A_X which are mapped to elements of Λ_X^+ under this bijection will be called "X-anti-dominant", and the set of those will be denoted by A_X^+.

2.2. **Knop's action.** F. Knop has defined an action of the Weyl group of $\mathbf{G}$ on the set of Borel orbits of maximal rank (cf. [Kn95]). We review it briefly: If $\mathbf{Y}$ is a $\mathbf{B}$-orbit and α is a simple root, then $\mathbf{Y}\mathbf{P}_\alpha/\mathcal{R}(\mathbf{P}_\alpha)$ is a homogeneous spherical variety for $\mathbf{P}_\alpha/\mathcal{R}(\mathbf{P}_\alpha) \simeq \mathbf{PGL}_2$, hence has one of the following types: Type G ($\mathbf{PGL}_2 \setminus \mathbf{PGL}_2$), type U ($\mathbf{SU}\backslash \mathbf{PGL}_2$ where $\mathbf{U}$ a maximal unipotent subgroup and $\mathbf{S} \subset \mathcal{N}(\mathbf{U})$), type T ($\mathbf{T}\backslash \mathbf{PGL}_2$, where $\mathbf{T}$ is a non-trivial torus) or type N ($\mathcal{N}(\mathbf{T})\backslash \mathbf{PGL}_2$). We say, correspondingly, that the pair (Y,α) is of type G, U, T or N. Since we are working over a non-algebraically closed field, in case T we will distinguish two sub-cases, called "split" and "non-split", according to whether $\mathbf{T}$ has the corresponding property. By definition, the simple reflection w_α fixes the orbit of maximal rank in $\mathbf{Y}\mathbf{P}_\alpha$, unless it is of type U, in which case there are two orbits of maximal rank and w_α interchanges them. The action defined this way is transitive on the set of Borel orbits of maximal rank (which includes the open $\mathbf{B}$-orbit). The stabilizer of the open orbit is $W_{(X)} := W_X \ltimes W_{P(X)}$, where W_X is a canonical subgroup of W called the "little Weyl group" of $\mathbf{X}$. The group $W_{(X)}$ normalizes $\mathcal{X}(\mathbf{X})$ and acts on it through the quotient W_X. It is known that the dual action of W_X on $\mathcal{Q}$ is faithful, generated by reflections, and admits the cone $\mathcal{V}$ defined above as a fundamental domain. (More on this action will be recalled in §6.1.)

Since we are also discussing non-trivial line bundles $\mathcal{L}_\Psi$ as were described in §1.1, there is also a fifth case, called (U,ψ). For a discussion of how this fits into the same setting (more precisely, into Knop's extension of his action to non-spherical varieties) we refer the reader to [Sa08, §5.5]. We caution the reader that, while the little Weyl group is still well-defined in this case, it does not coincide with the little Weyl group of the spherical variety. In this case the action of Knop is not transitive on all orbits of maximal rank, but only on some, called "admissible".

2.3. **Brion's description of simple spherical reflections.** Consider the labelled graph $\mathfrak{G}$ (to be called "Knop's graph") whose vertices are the $\mathbf{B}$-orbits of maximal rank, and where between any two vertices corresponding to orbits $\mathbf{Y}$ and $\mathbf{Z}$ there is an edge labelled by a simple root α if ${}^{w_\alpha}\mathbf{Y} = \mathbf{Z}$. Let $\mathbf{Y}$ be an orbit of maximal rank. Following Brion [Br01], for every path γ from $\mathbf{Y}$ to $\mathring{\mathbf{X}}$ we denote by $w(\gamma)$ the Weyl group element corresponding to the path, having the property that ${}^{w(\gamma)}\mathbf{Y} = \mathring{\mathbf{X}}$, and by $W(\mathbf{Y})$ the set of all $w(\gamma)$, where γ is a path between $\mathbf{Y}$ and $\mathring{\mathbf{X}}$ of length equal to the codimension of $\mathbf{Y}$. Let also $\mathfrak{G}(\mathbf{Y})$ denote the set of such paths. Then we have:

2.3.1. **Proposition** ([Br01], Propositions 2 and 4). *Given $w, w' \in W(\mathbf{Y})$ there is a series $w = w_0, w_1, \dots, w_n = w'$ in $W(\mathbf{Y})$ such that any consecutive w_i, w_{i+1} can be written as $w_i = uw_\alpha v$, $w_{i+1} = uw_\beta v$ with $l(w_i) = l(w_{i+1}) = l(u) + l(v) + 1$ and α, β two mutually orthogonal simple roots.*

Based on this, Brion proves:

2.3.2. **Theorem** ([Br01], Theorem 4). *A set of generators of $W_{(X)}$ consists of the elements w_α, $\alpha \in \Delta(\mathbf{X})$, and elements w with the following property: There is a decomposition $w = w_1^{-1} w_2 w_1$ such that:*

- ${}^{w_1}\mathring{X} =: Y$ *with* $\operatorname{codim}(Y) = l(w_1)$ *(i.e.* $w_1^{-1} \in W(\mathbf{Y})$*.)*
- w_2 *is either of the following two:*
  (1) *equal to* w_α*, where* α *is a simple root such that* (Y, α) *is of type* T *or* N*, or*
  (2) *equal to* $w_\alpha w_\beta$*, where* α, β *are two orthogonal simple roots which both lower* Y *to the same orbit* Y'*.*

(This description does not apply to the case of a non-trivial $\mathcal{L}_\Psi$ and the extension of Knop's action there.)

It follows from a theorem that we will recall later (Theorem 3.7.1) that the generators w of this theorem which are not of the form $w_\alpha, \alpha \in \Delta(\mathbf{X})$ all belong to the little Weyl group W_X. We will also see in §6.1 that the "canonical" generators for W_X, the simple reflections corresponding to "spherical roots", admit a description of this form.

2.4. **Hyperspecial and Iwahori orbits.** As mentioned, we assume that $\mathbf{G}$ possesses a smooth model over $\mathfrak{o}$, so that $K = \mathbf{G}(\mathfrak{o})$ is a hyperspecial maximal compact subgroup. We also set J for the Iwahori subgroup of K, that is, the inverse image of the Borel subgroup under the map $K \to \mathbf{G}(\mathbb{F}_q)$. We will assume throughout the following axioms pertaining to K- and J-orbits on X.

2.4.1. **Axiom.** *The set* A_X^+ *contains a complete set of representatives for* K*-orbits on* X*; elements of* A_X^+ *which map to distinct elements of* Λ_X^+ *belong to different* K*-orbits.*

2.4.2. **Axiom.** *For* $x \in A_X^+$ *we have* $xJ \subset x\mathbf{B}(\mathfrak{o})$*.*

The first axiom generalizes the Iwasawa (for $\mathbf{X} = \mathbf{U}\backslash\mathbf{G}$) and Cartan (for $\mathbf{X} = \mathbf{G}'$, $\mathbf{G} = \mathbf{G}' \times \mathbf{G}'$) decompositions. It was proven by Luna and Vust [LV83] in the case $\mathfrak{o} = \mathbb{C}[[t]]$. An alternate proof was given by Gaitsgory and Nadler in [GN], which can be adapted, with minor modifications, to the p-adic case. The adaptation requires the existence and a certain property, over $\mathfrak{o}$, of certain compactifications of the spherical variety which, if the spherical variety is defined and the group is split over a global field, are satisfied at almost every place. The second axiom also follows from the existence and local structure of such compactifications. I hope to include a complete presentation of this in future work; for now, the reader may take the above as axioms and verify them in every particular case. In the case of a symmetric variety, alternate proofs of similar statements have recently been presented by Benoist and Oh [BO07], Delorme and Sécherre [DS].

Remark. From the first axiom it follows that we have surjective maps: $A_X^+/A_0 \to X/K \to \Lambda_X^+$, but in general, unlike the case $\mathfrak{o} = \mathbb{C}[[t]]$, these maps are not bijective. An example is: $\mathbf{G} = \mathbb{G}_m$, $\mathbf{X} = \{\pm 1\}\backslash\mathbf{G}$ where the first map is bijective but not the second (since the residue field has non-trivial square classes), and another is $\mathbf{X} = \mathbf{O}_2\backslash\,\mathbf{GL}_2$, the space of non-degenerate quadratic forms (assume that the residue characteristic is not two), where the first map is also not bijective since the set $\mathbf{A}_X(\mathfrak{o})/\mathbf{A}(\mathfrak{o})$ has four elements, but there are only two classes of non-degenerate quadratic forms over the residue field, and hence these four elements are contained in only two orbits of K on X. (We certainly expect that the second map is bijective when $\mathbf{H}$ is connected.)

3. Intertwining operators

3.1. **Overview.** The goal of this paper is to compute eigenfunctions of the Hecke algebra on $C^\infty(X)$ or, more generally, on $C^\infty(X,\mathcal{L}_\Psi)$, when L_Ψ is as in §1.1[3]. We let $I(\chi)=\mathrm{Ind}_B^G(\chi\delta^{\frac12})$, a principal series representation. Then a Hecke eigenfunction on $C^\infty(X)$ is the image of a K-invariant vector $\phi_{K,\chi}\in I(\chi)$ via an intertwining operator $I(\chi)\to C^\infty(X)$ (for some χ).

The details of the present section are quite technical, therefore we give here an overview of its contents. We will introduce a G-eigenmeasure $|\omega_X|$ on X to set up a duality:

$$\begin{aligned} C_c^\infty(X)\otimes\nu\otimes C^\infty(X)&\to\mathbb{C} \\ \phi\otimes|\omega_X|\otimes f&\mapsto\int_X \phi f|\omega_X| \end{aligned} \tag{3.1}$$

(where ν is the character of $|\omega_X|$). As notation suggests, ω_X is the absolute value of a volume eigen-form on $\mathbf{X}$, the eigencharacter of which will be denoted by $\mathfrak{n}$ (hence $\nu=|\mathfrak{n}|$).

For every B-orbit Y on X (assuming for now that $Y=\mathbf{Y}(k)$ is a single B-orbit) we will introduce morphisms:

$$S_\chi^Y: C_c^\infty(X)\to I(\chi)$$

given (in some domain of convergence for χ) by the formula:

$$S_\chi^Y(\phi)(1)=\int_{\bar{Y}}\phi\mu_\chi^Y$$

where μ_χ^Y is a suitable B-eigenmeasure on $\bar{Y}$. For $\mathbf{Y}=\mathring{\mathbf{X}}$ we will sometimes omit the exponent ^{Y} from the notation.

We view $I(\chi)$ as a subspace of the space of smooth, *tempered* generalized functions on $U\backslash G$, and through the duality (3.1) we have an adjoint for $S^Y_{\chi^{-1}\nu^{-1}}$ (again, by fixing an invariant measure on $U\backslash G$):

$$\Delta_\chi^Y:\mathcal{S}(U\backslash G)\to C^\infty(X).$$

The space $\mathcal{S}(U\backslash G)$ is a larger space than $C_c^\infty(U\backslash G)$ suitable for the theory of Fourier transforms, called the *Schwartz space of* $\overline{U\backslash G}^{\mathrm{aff}}$. (The exponent aff denotes the *affine closure* of $U\backslash G$; however we will allow ourselves to abuse language and notation, write $\mathcal{S}(U\backslash G)$ and say "Schwartz space of $U\backslash G$".)

We will show that Δ_χ^Y can be expressed (in a suitable domain of convergence for χ) by a formula:

$$\Delta_\chi^Y(\Phi)(\xi)=\int_{U\backslash G}\Phi\mu_\chi'^Y$$

where the point of evaluation ξ belongs to Y and $\mu_\chi'^Y$ is a suitable $A\times G_\xi$-eigenmeasure on $U\backslash G$. (Notice that $U\backslash G$ carries a natural action of $A\times G$.)

While our goal is to compute $\Delta_\chi^Y(\Phi_K)$, with $\Phi_K\in\mathcal{S}(X)^K$, this cannot be done directly. Rather, we first compute $\Delta_\chi^Y(\Phi_J)$, where Φ_J is a suitable function of small support in $\mathcal{S}(U\backslash G)^J$, and explain the steps needed to deduce from this

[3]The case of $\mathcal{L}_\Psi$ will generally be suppressed from our notation, except when it needs to be treated separately. Unless otherwise stated, all our statements hold – with obvious modifications – in that case, as well.

the computation of $\Delta^Y_\chi(\Phi_K)$. These steps will be undertaken in the following two sections.

The biggest part of this section is devoted in choosing eigenmeasures in compatible ways and rigorously proving the formula for Δ^Y_χ. A major complication that arises is that $Y = \mathbf{Y}(k)$ will, in general, consist of several $\mathbf{B}$-orbits – in this case the morphisms Δ^Y_χ are not uniquely defined, and we must instead introduce variants, denoted $\Delta^Y_{\tilde\chi}$. These are defined by suitable eigenmeasures on Y, and the correct way to choose eigenmeasures is that these be absolute values of differential eigenforms. Then we need to compare differential forms on different $\mathbf{B}$-orbits with each other, or with differential forms on $\mathbf{U}\backslash\mathbf{G}$. The most technical parts of this section can be skipped at first reading.

3.2. **Morphisms.** Let $\mathbf{Y}$ be a $\mathbf{B}$-orbit on $\mathbf{X}$ with a distinguished k-point y_0 (we will later discuss how to choose y_0), and let $\mathbf{A}_Y$ denote the quotient of $\mathbf{A}$ by the stabilizer modulo $\mathbf{U}$ of y_0. Fix a top-degree $\mathbf{B}$-eigenform ω_Y on $\mathbf{Y}$, whose character we will denote by $\mathfrak{c}$, and let $\tilde\chi'$ vary over all eigencharacters of A_Y which are unramified when restricted to the image of $A \to A_Y$. Through our choice of y_0 we get an identification: $\mathbf{Y}/\mathbf{U} \simeq \mathbf{A}_Y$, and we consider $\tilde\chi'$ as a function on Y. In [Sa08] we defined a morphism $S^Y_{\tilde\chi} : C_c^\infty(X) \to I(\chi)$ which, composed with evaluation at "1" is given by the rational continuation[4] of the integral:

$$\mathrm{ev}_1 \circ S^Y_{\tilde\chi}(\phi) = \int_Y \phi(y)\tilde\chi'^{-1}(y)|\omega_Y|(y). \tag{3.2}$$

3.2.1. **Definition.** An eigenmeasure on Y of the form $\tilde\chi'^{-1}|\omega_Y|$ will be called *pseudo-rational*.

We explain now what the index $\tilde\chi$ which appears in the notation is: Consider the map: $\mathbf{A}(\bar k) \to \mathbf{A}_Y(\bar k)$ and let R denote the preimage of $\mathbf{A}_Y(k)$. The expression (3.2) provides a complex character $\tilde\chi$ of R defined as: $\tilde\chi = \delta^{-\frac12} \cdot \tilde\chi' \cdot |\mathfrak{c}|^{-1}$. (Notice that unramified characters such as $\delta^{-\frac12}$ and $|\mathfrak{c}|^{-1}$ make sense on the whole of $\mathbf{A}(\bar k)$ – the reason is that there are canonical algebraic characters, of which these unramified characters are a power of the absolute value.) The characters obtained this way are a torsor for the group of characters of A_Y which are unramified on the image of A. If χ denotes the restriction of $\tilde\chi$ to A, the above expression (3.2) defines a $(B, \chi\delta^{\frac12})$-equivariant functional on $C_c^\infty(X)$, and by Frobenius reciprocity a morphism: $C_c^\infty(X) \to I(\chi)$. When no confusion arises, we will denote by $S^Y_{\tilde\chi}$ both the functional and the corresponding morphism into $I(\chi)$.

For the principal series $I(\chi)$ to admit such a morphism, the character χ must satisfy the condition:

$$\chi^{-1}\delta^{\frac12}|_{B_y} = \delta_{B_y} \tag{3.3}$$

for a (any) point $y \in Y$. By abuse of language we will often say that "$\tilde\chi$ is a character of A_Y which extends χ", although neither $\tilde\chi$ nor χ are, in general, characters of A_Y.

[4]It was shown that this integral converges and represents a rational function for a certain domain of the parameter χ. We will discuss again domains of convergence in §7.3.

Notice that until now our definition of the family $S^Y_{\check\chi}$ depends on the choice of y_0 and of ω_Y. This was enough for the purposes of [Sa08], but here we need to be more careful and to normalize the morphisms $S^Y_{\check\chi}$ in a precise way.

3.3. **Compatible choices of eigenforms and B_0-orbits.** Our goal now is to normalize the morphisms $S^Y_{\check\chi}$ of (3.2) in compatible ways, for all $\mathbf{B}$-orbits $\mathbf{Y}$ of maximal rank. At the same time, we will also define compatible isomorphisms: $\mathbf{Y}/\mathbf{U} \simeq \mathbf{A}_Y$, up to multiplication by A_0.

3.3.1. **Lemma.** *Given a $\mathbf{B}$-orbit $\mathbf{Y}$ with k-points, the following data are equivalent:*

(1) *A trivialization of the $\mathbf{A}_Y$-torsor $\mathbf{Y}/\mathbf{U}$, i.e. an isomorphism: $\mathbf{Y}/\mathbf{U} \simeq \mathbf{A}_Y$.*
(2) *A splitting of the short exact sequence:*

$$1 \to k^\times \to k(\mathbf{Y})^{(\mathbf{B})} \to \mathcal{X}(\mathbf{Y}) \to 1.$$

(3) *A family $\{\omega_{\mathfrak{c}}\}_{\mathfrak{c}}$ of non-zero k-rational $\mathbf{B}$-eigen-volume forms on $\mathbf{Y}$, determined up to a common multiple, where $\mathfrak{c}$ ranges over all possible characters of such eigenforms and the $\omega_{\mathfrak{c}}$ have the property that their quotients (which are $\mathbf{B}$-eigenfunctions) all are equal to 1 at the same k-point of Y.*

Proof. The implication $1 \Rightarrow 2$ is obvious. For the converse, we notice that $k(\mathbf{Y})^{(\mathbf{B})} = k[\mathbf{Y}]^{(\mathbf{B})}$ and $k[\mathbf{Y}/\mathbf{U}] = \oplus_{\chi\in\mathcal{X}(\mathbf{Y})} k[\mathbf{Y}]_\chi$, where $k[\mathbf{Y}]_{\mathcal{X}}$ denotes the corresponding 1-dimensional subspace. For any splitting $\chi \mapsto e_\chi \in k[\mathbf{Y}]_\chi$ the maps $e_\chi \mapsto 1 \in k$ extend uniquely to a homomorphism of algebras: $k[\mathbf{Y}/\mathbf{U}] \to k$, i.e. a k-point on $\mathbf{Y}/\mathbf{U}$.

For the third condition we notice that there exists an eigen-volume form on $\mathbf{Y}$ with character $\mathfrak{c}$ if and only if $\mathfrak{c}|_{\mathbf{B}_y} = \mathfrak{d}_{\mathbf{B}_y}\mathfrak{d}_{\mathbf{B}}^{-1}|_{\mathbf{B}_y}$, and that every algebraic character of $\mathbf{B}_y$ (where $y \in \mathbf{Y}(k)$) extends to a character of $\mathbf{B}$. In particular, there exists an eigen-volume form. Multiplying such a form by elements of $\mathcal{X}(\mathbf{Y})$, considered as eigenfunctions on $\mathbf{Y}$ via an identification $\mathbf{Y}/\mathbf{U} \simeq \mathbf{A}_Y$ we get a family $\{\omega_{\mathfrak{c}}\}_{\mathfrak{c}}$ with the stated property, which depends only up to a common scalar on the form chosen originally. Vice versa, the quotients of elements in such a family are eigenfunctions distinguishing a unique point in $\mathbf{Y}/\mathbf{U}(k)$. ∎

A variant of this lemma is:

3.3.2. **Lemma.** *Given a $\mathbf{B}$-orbit $\mathbf{Y}$, the following data are equivalent:*

(1) *An isomorphism: $Y/A_0U \simeq A_Y/A_0$.*
(2) *A multiplicatively-closed family of non-zero A_Y-eigenfunctions on Y/U, one for each character of A_Y which is unramified in the image of $A \to A_Y$.*
(3) *A family of non-zero pseudo-rational B-eigenmeasures on Y, determined up to a common multiple, which have the property that their quotients by any fixed element of the family form a family of eigenfunctions as in 2.*

Notice also that choosing an isomorphism: $A_Y/A_0 \to Y/A_0U$ with $1 \mapsto$ (the coset of an element of $\mathbf{Y}(\mathfrak{o})$) is equivalent to choosing a B_0-orbit on $\mathbf{Y}(\mathfrak{o})$.

3.3.3. *Generalities on differential forms.* Let $\mathbf{G}_1 \supset \mathbf{G}_2 \supset \mathbf{G}_3$ be algebraic groups. The following is the algebro-geometric version of the factorization of an integral:

$$\int_{G_3\backslash G_1} f(g)dg = \int_{G_2\backslash G_1}\int_{G_3\backslash G_2} f(hx)dhdx$$

(where dg, dh, dx are suitable eigenmeasures), which will be useful when the corresponding sets of k-points do not surject on the k-points of the quotient.

Let $\mathfrak{q} : \mathbf{V} \to \mathbf{Y}$ be a smooth morphism of schemes. Then, by definition, the sheaf of relative differentials $\Omega_{V/Y}$ is locally free on $\mathbf{V}$. Now assume in addition that $\mathbf{Y}$ (and hence also $\mathbf{V}$) is smooth over spec(k), and let Λ^{top} denote the top-degree exterior powers (i.e. determinants) of the vector bundles $\Omega_V, \Omega_Y, \Omega_{V/Y}$. From the short exact sequence: $0 \to \Omega_Y \xrightarrow{\mathfrak{q}^*} \Omega_V \to \Omega_{V/Y} \to 0$ we get a canonical isomorphism of line bundles on $\mathbf{V}$: $\Lambda^{\text{top}}(\Omega_V) = \Lambda^{\text{top}}(\Omega_{V/Y}) \otimes \mathfrak{q}^* \Lambda^{\text{top}}(\Omega_Y)$. If ω_1 denotes a section of $\Lambda^{\text{top}}(\Omega_{V/Y})$ and ω_2 denotes a section of $\Lambda^{\text{top}}(\Omega_Y)$ the corresponding section ω_3 of $\Lambda^{\text{top}}(\Omega_V)$ will be denoted (by a slight abuse of notation) by $\omega_1 \wedge \mathfrak{q}^*(\omega_2)$ and will be called a *factorization* of ω_3 with respect to the morphism $\mathfrak{q} : \mathbf{V} \to \mathbf{Y}$.

Now we return to the situation of $\mathbf{G}_1 \supset \mathbf{G}_2 \supset \mathbf{G}_3$. Let ω be a $\mathbf{G}_1$-eigen-volume form on $\mathbf{G}_3 \backslash \mathbf{G}_1$. For a non-zero such form to exist, the character $\mathfrak{c}$ of ω must satisfy: $\mathfrak{c}|_{\mathbf{G}_3} = \mathfrak{d}_{\mathbf{G}_3} \mathfrak{d}_{\mathbf{G}_1}^{-1}|_{\mathbf{G}_3}$. Let $\mathfrak{c}'$ be the character $\mathfrak{c}\mathfrak{d}_{\mathbf{G}_2}^{-1}\mathfrak{d}_{\mathbf{G}_1}|_{\mathbf{G}_2}$ of $\mathbf{G}_2$. Any character $\mathfrak{c}'$ of $\mathbf{G}_2$ defines a line bundle over $\mathbf{G}_2 \backslash \mathbf{G}_1$, equipped with a trivialization of its pull-back to $\mathbf{G}_1$; by definition, the sections over an open $\mathbf{Y} \subset \mathbf{G}_2 \backslash \mathbf{G}_1$ with preimage $\mathbf{V} \subset \mathbf{G}_1$ are sections f of $\mathfrak{o}_V$ such that $f(g_2 g) = \mathfrak{c}'(g_2) f(g)$ for any $g_2 \in \mathbf{G}_2$. Let $\mathcal{L}$ be the line bundle defined by $\mathfrak{c}'$. Then ω admits a factorization: $\omega_1 \wedge \mathfrak{q}^*(\omega_2)$ with respect to $\mathfrak{q} : \mathbf{G}_3 \backslash \mathbf{G}_1 \to \mathbf{G}_2 \backslash \mathbf{G}_1$ and $\mathcal{L}$, where $\omega_1 \in \Gamma(\mathbf{G}_3 \backslash \mathbf{G}_1, \Lambda^{\text{top}}(\Omega_{(G_3 \backslash G_1)/(G_2 \backslash G_1)}) \otimes q^* \mathcal{L}^{-1})$ and $\omega_2 \in \Gamma(\mathbf{G}_2 \backslash \mathbf{G}_1, \Lambda^{\text{top}}(\Omega_{G_2 \backslash G_1}) \otimes \mathcal{L})$ are eigenforms for $\mathbf{G}_1$ and $\mathbf{G}_2$, with eigencharacters $\mathfrak{c}$ and $\mathfrak{c}'$, respectively. For every point $x \in \mathbf{G}_3 \backslash \mathbf{G}_1$, the form ω_1 gives rise to a volume form on the $\mathbf{G}_2$-orbit of x: First, the choice of x gives rise to a trivialization of $\mathfrak{q}^* \mathcal{L}^{-1}$ over its $\mathbf{G}_2$-orbit, and then the pull-back of ω_1 via the inclusion $x\mathbf{G}_2 \to \mathbf{G}_3 \backslash \mathbf{G}_1$ is a volume form on $x\mathbf{G}_2$. Therefore, the form ω_1 will be called a "volume form along $\mathbf{G}_2$-orbits".

3.3.4. *The case of two adjacent orbits.* Now let $\mathbf{Y}$ and $\mathbf{Z}$ be two $\mathbf{B}$-orbits on $\mathbf{X}$ with $\mathbf{Z} \subset \overline{\mathbf{Y}}$, α a simple root joining them of type U under Knop's action. Assume that we are given a family of non-zero pseudo-rational B-eigenmeasures on Y as in Lemma 3.3.2. Assume moreover that the distinguished $A_0 U$ orbit on Y is that of a point $y_0 \in \mathbf{Y}(\mathfrak{o})$. We will show how to associate to this family a similar family of eigenforms on $\mathbf{Z}$, uniquely up to the action of A_0.

Pick a representative $\tilde{w}_\alpha$ for w_α in $\mathcal{N}(\mathbf{A})(\mathfrak{o})$; then as schemes: $\mathbf{Z} \times \mathbf{U}_\alpha \simeq \mathbf{Y}$ under the map $(z, u) \mapsto z\tilde{w}_\alpha u$. Thus, to a $\mathbf{B}$-eigen-volume form ω_Z on $\mathbf{Z}$ (with eigencharacter $\mathfrak{c}$) we can associate the volume form $\omega_Y = \omega_Z \wedge dx$ on $\mathbf{Y}$, where dx denotes the obvious eigenform on $\mathbf{U}_\alpha$ induced through an $\mathfrak{o}$-isomorphism: $\mathbf{U}_\alpha \simeq \mathbb{A}^1_x$. Then ω_Y is also an eigenform for $\mathbf{B}$ with eigencharacter equal to $e^{-\alpha} \cdot {}^{w_\alpha}\mathfrak{c}$. Notice that the corresponding measure $|\omega_Y|$ does not depend on the choices made. Hence, for any family of pseudo-rational eigenmeasures on Y as in Lemma 3.3.2.(3), we get a family of pseudo-rational eigenmeasures on Z.

This process, in particular, gives rise by Lemma 3.3.2 to compatible choices of B_0-orbits on $\mathbf{Y}(\mathfrak{o})$ and $\mathbf{Z}(\mathfrak{o})$ which we explicate here: Let $y_0 B_0$ be a given B_0-orbit on $\mathbf{Y}(\mathfrak{o})$, consider the quotient $\mathbf{Y}\mathbf{P}_\alpha \to \mathbf{Y}\mathbf{P}_\alpha / \mathbf{U}_{P_\alpha}$ and denote by $\overline{y_0 B_0}$ the image of $y_0 B_0$. The $\mathbf{L}_\alpha(\mathfrak{o})$-orbit of that is isomorphic to $F\mathbf{U}_\alpha(\mathfrak{o}) \backslash \mathbf{L}_\alpha(\mathfrak{o})$, where $F \subset \mathbf{B}_\alpha(\mathfrak{o})$ a finite subgroup. Let J_α be the Iwahori subgroup of $\mathbf{L}_\alpha$ with respect to the Borel $\mathbf{L}_\alpha \cap \mathbf{B}$. By the Iwahori-Bruhat decomposition, $\mathbf{L}_\alpha(\mathfrak{o}) = J_\alpha \sqcup J_\alpha w_\alpha J_\alpha = J_\alpha \sqcup \mathbf{U}_\alpha(\mathfrak{o}) w_\alpha \mathbf{B}_\alpha(\mathfrak{o})$, the group J_α has two orbits in $\bar{y}_0 \mathbf{L}_\alpha(\mathfrak{o})$: one of them is the

$\mathbf{B}_\alpha(\mathfrak{o})$-orbit of y_0 and the other intersects the smaller $\mathbf{B}_\alpha$-orbit in a unique $\mathbf{B}_\alpha(\mathfrak{o})$-orbit $\overline{z_0 B_0}$. Each fiber of the quotient: $\mathbf{YP}_\alpha \to \mathbf{YP}_\alpha/\mathbf{U}_{P_\alpha}$ being homogeneous under the unipotent group $\mathbf{U}_{P_\alpha}$, the preimage of $\overline{z_0 B_0}$ intersects $\mathbf{Z}(\mathfrak{o})$ in a unique B_0-orbit $z_0 B_0$.

3.3.5. **Lemma.** *Let* $\mathbf{Y}$, $\mathbf{Z}$ *be two orbits joined by a root* α *under Knop's action, and assume that they are equipped with compatible families of pseudo-rational* B*-eigenmeasures as above. Use these eigenmeasures to define the morphisms* $S^Y_{\tilde{\chi}}$ *of (3.2). Let* $T_{w_\alpha} : I(\chi) \to I({}^{w_\alpha}\chi)$ *denote the standard intertwining operator between principal series:*

$$T_{w_\alpha}(f)(g) = \int_{U_\alpha} f(\tilde{w}_\alpha u g) du$$

defined with some preimage $\tilde{w}_\alpha \in \mathcal{N}(\mathbf{A})(\mathfrak{o})$ *of* w_α*, and with measure* dx *on* $U_\alpha \simeq \mathbb{A}^1_x$ *(with the latter isomorphism over* $\mathfrak{o}$*). Then:*

$$T_{w_\alpha} \circ S^Z_{\tilde{\chi}} = S^Y_{{}^{w_\alpha}\tilde{\chi}}.$$

This is just a strengthened version of a special case of [Sa08, Theorem 5.2.1].

Proof. We know already from [Sa08, Theorem 5.2.1] that one side has to be a rational multiple of the other. To compute the proportionality constant, it is enough to do it when the morphisms of both sides – viewed as functionals by Frobenius reciprocity, are applied to functions with support in $Z \cup Y$. For those, one writes down explicitly the integral expressions for both sides, when $\tilde{\chi}$ is in the region of convergence of T_{w_α}. ☐

Now consider Knop's graph $\mathfrak{G}$, which was defined in §2.3. We fix a class of pseudo-rational eigenmeasures, to be called "standard", on $\mathring{X}$, with the property that their quotients are equal to 1 on our distinguished point x_0. This condition determines them up to a common multiple, and the precise normalization will be discussed in a later chapter. Choosing a path γ in $\mathfrak{G}$ joining $\mathring{X}$ to Y, the above process induces a similar class of eigenmeasures on Y (and hence also a B_0-orbit $y_0 B_0 \subset \mathbf{Y}(\mathfrak{o})$) which, a priori, depends on the choice of path γ. Our goal is to show:

3.3.6. **Proposition.** *The induced class of eigenmeasures on* Y *(and therefore also the orbit* $y_0 B_0$*) does not depend on the choice of path* γ*.*

Proof. First, we claim that the induced class of eigenmeasures on Y does not depend on γ itself but only, possibly, on $w(\gamma)$. For this, it is enough to show that for two paths $\gamma_1, \gamma_2 \in \mathfrak{G}(\mathbf{Y})$ with $w(\gamma_1) = w(\gamma_2) = w$ the families of morphisms $S^Y_{\tilde{\chi}}$ defined with the eigenmeasures obtained through each of the two paths coincide. But this follows from Lemma 3.3.5: Indeed, the lemma implies that $T_w \circ S^Y_{\tilde{\chi}} = S^{\mathring{X}}_{{}^w\tilde{\chi}}$ for both families $S^Y_{\tilde{\chi}}$, and therefore $S^Y_{\tilde{\chi}}$ is the same rational multiple of $T_{w^{-1}} \circ S^{\mathring{X}}_{{}^w\tilde{\chi}}$ for both families.

Now, using Proposition 2.3.1, it is sufficient to prove the following: If α, β are two orthogonal simple roots which raise an orbit $\mathbf{Z}$ to an orbit $\mathbf{Y}$ then the identifications defined by α and β coincide. We can substitute the variety $\mathbf{YP}_{\alpha\beta}$ by the variety $\mathbf{YP}_{\alpha\beta}/\mathbf{U}_{P_{\alpha\beta}}$. The $\mathbf{L}'_{\alpha\beta}$-orbit of a $\mathfrak{o}$-point on the latter is isomorphic to one of the varieties $\mathbf{SL}_2$, $\mathbf{PGL}_2$, acted upon by left and right multiplication (under an isomorphism $\mathbf{L}'_{\alpha\beta} \simeq \mathbf{SL}_2$). We are thus reduced to the case $\mathbf{Y}$ = the open Bruhat cell and $\mathbf{Z}$ = the closed Bruhat cell in $\mathbf{SL}_2$ (or $\mathbf{PGL}_2$). Then the

isomorphisms $\mathbf{Z}\times\mathbf{U}_\alpha \simeq \mathbf{Y}$ and $\mathbf{Z}\times\mathbf{U}_\beta\simeq\mathbf{Y}$, used in §3.3.4 to define compatible choices of eigenmeasures, correspond to nothing else than the decomposition of the open Bruhat cell as a product $\mathbf{U}w\mathbf{B}$ or $\mathbf{B}w\mathbf{U}$, respectively, and hence it is immediate to see that given a class of eigenforms on $\mathbf{B}$ of the form of Lemma 3.3.2, the induced class of eigenforms on $\mathbf{B}w\mathbf{B}$ does not depend on which decomposition we use. □

This proposition allows us to define the notion of "standard" eigenmeasures on each orbit Y of maximal rank, which depends only on the choice of standard eigenmeasures on $\mathring{X}$.

3.4. **Adjoints.** Having chosen a point $x_0\in\mathring{\mathbf{X}}(\mathfrak{o})$, we normalize our family of standard pseudo-rational eigenmeasures on $\mathring{X}$ by requiring that $\mu(x_0J)=1$, where J is the Iwahori subgroup. Recall that $x_0J\subset x_0B_0$ by Axiom 2.4.2, therefore such a normalization is possible, since all eigenmeasures in the family differ by unramified B-eigenfunctions. So we have:

3.4.1. **Definition.** The pseudo-rational B-eigenmeasures on $\mathring{X}$ normalized according to the property $\mu(x_0J)=1$ will be called *standard*. The corresponding eigenmeasures on any orbit Y of maximal rank, obtained via Proposition 3.3.6, will also be called *standard*.

From now on we will only be using the morphisms $S^Y_{\tilde\chi}$ defined as in (3.2) with the use of *standard* eigenforms and eigenmeasures.

Recall that we are assuming the existence of a G-eigenmeasure on X with unramified eigencharacter ν. This measure is the absolute value of a volume eigenform ω_X on $\mathring{\mathbf{X}}$, which from now on we will also assume to be in the class of standard eigenforms. Multiplication by this measure yields an identification:

$$C_c^\infty(X)\otimes\nu\simeq M_c^\infty(X). \tag{3.4}$$

We can now, via (3.4), modify $S^Y_{\tilde\chi^{-1}\nu^{-1}}$ into a morphism:

$$\tilde S^Y_{\tilde\chi^{-1}\nu^{-1}}: M_c^\infty(X)\simeq C_c^\infty(X)\otimes\nu\xrightarrow{S^Y_{\tilde\chi^{-1}\nu^{-1}}\otimes 1} I(\chi^{-1}\nu^{-1})\otimes\nu\simeq I(\chi^{-1})$$

via the canonical isomorphism: $I(\chi^{-1}\nu^{-1})\otimes\nu\simeq I(\chi^{-1})$. From now on we drop the notation $\tilde S^Y_{\tilde\chi^{-1}\nu^{-1}}$, and we will denote $\tilde S^Y_{\tilde\chi^{-1}\nu^{-1}}$ by $S^Y_{\tilde\chi^{-1}\nu^{-1}}$. It should be clear from the context whether we are refering to a morphism from $C_c^\infty(X)$ to $I(\chi^{-1}\nu^{-1})$ or a morphism from $M_c^\infty(X)$ to $I(\chi^{-1})$.

Remark. In fact, it is more natural to introduce the morphism $\tilde S_{\tilde\chi^{-1}\nu^{-1}}$ (recall that we have $\mathbf{Y}=\mathring{\mathbf{X}}$ when the exponent ^{Y} is omitted) directly as a morphism: $M_c^\infty(X)\to I(\chi^{-1})$, namely, integration of any measure against the character $\delta^{-\frac12}\tilde\chi$, considered as a function on $\mathring{X}$ via the choice of the point x_0. The reference to $C_c^\infty(X)$ is artificial and less canonical, since it depends on the choice of $|\omega_X|$. However, to handle the sheaf M_c^∞ on X and describe its restrictions to the smaller orbits, it is convenient to work with functions instead of measures, since restrictions of functions make sense as functions.

We will denote by

$$\Delta^Y_{\tilde\chi}: I(\chi)\to C^\infty(X)$$

the morphism adjoint to $S^Y_{\tilde\chi^{-1}\nu^{-1}}$. (Here we are using the standard pairing between $I(\chi)$ and $I(\chi^{-1})$: $(f_1,f_2)\mapsto\int_K f_1(k)f_2(k)dk$ with $\mathrm{Vol}(K)=1$.) Notice that for $Y=\mathring{X}$, $\Delta^Y_{\tilde\chi}$ is defined whenever $\chi\in\delta^{\frac12}A^*_X$. By our assumption that $\mathring{X}$ carries a B-invariant measure, we have $\delta|_{B_y}=\delta_{B_y}$ for a point y on the open orbit. It is known [BLV86, Théorème 3.4] that the unipotent radical of $\mathbf{B}_y$ is equal to $\mathbf{L}(\mathbf{X})\cap\mathbf{U}$; therefore, $\delta_{B_y}=\delta_{(X)}|_{B_y}$ and hence:

$$\delta_{P(X)}\in A^*_X \tag{3.5}$$

and the condition for the open orbit can be written:

$$\chi\in\delta^{\frac12}_{(X)}A^*_X. \tag{3.6}$$

In general, the condition (see (3.3)) is:

$$\chi\nu\delta^{\frac12}|_{B_y}=\delta_{B_y} \tag{3.7}$$

for any point $y\in Y$.

We want to obtain an explicit formula for $\mathrm{ev}_\xi\circ\Delta^Y_{\tilde\chi}$, where ξ is some point of Y. Assume for now that a point ξ has been chosen, and denote by $\mathbf{G}_\xi$ its stabilizer.

We fix volume eigen-forms on the spaces $\mathbf{X},\mathbf{G},\mathbf{U}\backslash\mathbf{G}$ as follows:

- As mentioned above, the volume form ω_X on $\mathbf{X}$ will be standard, hence normalized by the condition: $|\omega_X|(x_0J)=1$.
- On $\mathbf{U}\backslash\mathbf{G}$, we fix the invariant volume form $\omega_{U\backslash G}$ such that $|\omega_{U\backslash G}|(U\backslash UK)=1$.
- On $\mathbf{G}$ we fix the invariant volume form ω_G such that $|\omega_G|(K)=1$.

These induce differential forms ω_ξ, ω_U "along" $\mathbf{G}_\xi$-orbits, respectively $\mathbf{U}$-orbits, on $\mathbf{G}$ (cf. §3.3.3) which factorize ω_G with respect to the orbit maps:

$$o_\xi:\mathbf{G}\ni g\mapsto\xi g\in\mathbf{X}$$

and

$$o_U:\mathbf{G}\ni g\mapsto\mathbf{U}g\in\mathbf{U}\backslash\mathbf{G}.$$

In other words, in the notation of §3.3.3 we have:

$$\omega_G:=\mathfrak{n}^{-1}\omega_\xi\wedge o_\xi^*\omega_X=\omega_U\wedge o_U^*(\omega_{U\backslash G}).$$

(This statement is essentially just saying that for a compactly supported function ϕ on G we have:

$$\int_X\int_{G_x}(\nu^{-1}\phi)(hx)=\int_{U\backslash G}\int_U\phi(ug)$$

with respect to the corresponding measures.)

Finally, recall that we have a standard top $\mathbf{B}$-eigenform ω_Y on $\mathbf{Y}$, involved in the definition of $S^Y_{\tilde\chi}$ (3.2). We use ω_ξ to define compatible eigenforms $\omega_{\tilde Y},\omega_{\hat Y}$ on $\tilde{\mathbf{Y}}:=\mathbf{B}\mathbf{G}_\xi\subset\mathbf{G}$ and $\hat{\mathbf{Y}}:=\mathbf{U}\backslash\mathbf{B}\mathbf{G}_\xi\subset\mathbf{U}\backslash\mathbf{G}$; explicitly, these forms satisfy:

$$\omega_{\tilde Y}:=\iota^*(\omega_\xi\wedge o_\xi^*(\omega_Y))=\omega_U\wedge o_U^*(\omega_{\hat Y})$$

where $\iota:\mathbf{G}\to\mathbf{G}$ is the map $g\mapsto g^{-1}$. Notice that $\omega_{\tilde Y}$, resp. $\omega_{\hat Y}$, is a $\mathbf{B}\times\mathbf{G}_\xi$-eigenform, resp. an $\mathbf{A}\times\mathbf{G}_\xi$-eigenform, with eigencharacter equal to $\mathfrak{d}_{\mathbf{G}_\xi}$ times the character of ω_Y.

For reasons that will become apparent later, we would like to think of $I(\chi^{-1})$ as living in $C^\infty(U\backslash G)$ (more precisely, for almost every χ, in $C^\infty_{\mathrm{temp}}(U\backslash G)$, the space

of *tempered*, smooth generalized functions on $U\backslash G$, which will be defined later), and hence think of $\Delta^Y_{\tilde{\chi}}$ as a morphism:

$$M_c^\infty(U\backslash G) \to I(\chi) \to C^\infty_{\text{temp}}(X).$$

Identifying $M_c^\infty(U\backslash G)$ and $C_c^\infty(U\backslash G)$ via the invariant measure that we fixed before we get a morphism:

$$C_c^\infty(U\backslash G) \to I(\chi) \to C^\infty_{\text{temp}}(X),$$

still to be denoted by $\Delta^Y_{\tilde{\chi}}$. We notice that the first arrow of this morphism can be described as the integral:

$$\Phi \mapsto \int_A \Phi(Ua\bullet)\chi^{-1}\delta^{-\frac{1}{2}}(a)da$$

with measure 1 on A_0.

The following is an explicit description of $\Delta^Y_{\tilde{\chi}}$:

3.4.2. **Lemma.** *For $\phi \in C_c^\infty(U\backslash G)$, $\xi \in Y$ and χ in a suitable region of convergence we have:*

$$\text{ev}_\xi \circ \Delta^Y_{\tilde{\chi}}(\phi) = \int_{\hat{Y}} \phi(y)\nu\tilde{\chi}'^{-1}(y)|\omega_{\hat{Y}}|(y),$$

where $\tilde{\chi}'$ is the character of A_Y such that $\nu^{-1}\tilde{\chi}'|\mathfrak{c}|^{-1} = \delta^{\frac{1}{2}}\tilde{\chi}$, where $\mathfrak{c}$ is the eigencharacter of $\omega_{\hat{Y}}$ under the $\mathbf{A}$-action.

Here we regard ν as a function on $U\backslash G$, since it is U-invariant on G. The B-eigenfunction $\tilde{\chi}'$ on Y, pulled back to $\tilde{Y}$, can be considered as an $A\times G_\xi$-eigenfuntion on $\hat{Y} \subset U\backslash G$ with trivial eigencharacter for G_ξ. Notice that the eigencharacter of $\omega_{\hat{Y}}$ for the action of $\mathbf{G}_\xi$ is $\mathfrak{d}^{-1}_{\mathbf{G}_\xi}$, and $\nu|_{G_\xi} = \delta_{G_\xi}$, therefore the stated functional is indeed G_ξ-invariant.

Proof. We notice that with the chosen normalizations we have a commutative diagram of G-equivariant maps:

$$\begin{array}{ccc} C_c^\infty(G)\otimes\nu & \xrightarrow{o_{\xi*}} & C_c^\infty(X)\otimes\nu \\ \downarrow & & \downarrow \\ M_c^\infty(G) & \xrightarrow{o_{\xi*}} & M_c^\infty(X) \end{array}$$

where: the first vertical map is $\Phi \mapsto \Phi(g)\nu(g)|\omega_G|(g)$, the first horizontal map is integration over the fibers of o_ξ with respect to ω_ξ, the second horizontal map is push-forward of measures and the second vertical map is the one described previously, i.e. $\phi \mapsto \phi|\omega_X|$. A similar diagram holds for o_U (where the push-forward o_{U*} of functions is with respect to $|\omega_U|$) without tensoring with ν on the upper row.

Let $\Phi_1, \Phi_2 \in C_c^\infty(G)$ with images $o_{\xi*}(\Phi_1) = \phi_1 \in C_c^\infty(X)$, $o_{U*}(\Phi_2) = \phi_2 \in C_c^\infty(U\backslash G)$. (In particular, ϕ_1 is supported in $G_\xi\backslash G \subset X$.) Recall that, by definition

$S^Y_{\tilde\chi^{-1}\nu^{-1}}(\phi_1|\omega_X|) = \nu\cdot S_{\tilde\chi^{-1}\nu^{-1}}(\phi_1)$. We compute:

$$\left\langle \phi_1|\omega_X|, \Delta^Y_{\tilde\chi}\phi_2\right\rangle_X = \left\langle S^Y_{\tilde\chi^{-1}\nu^{-1}}(\phi_1|\omega_X|), \phi_2\right\rangle_{U\backslash G} =$$

$$= \int_{U\backslash G}\left(\nu(g)\int_Y \phi_1(yg)\tilde\chi'(y)|\omega_Y|(y)\right)\phi_2(g)|\omega_{U\backslash G}|(g) =$$

$$= \int_G \nu(g)\int_{\tilde Y}\Phi_1(z^{-1}g)\Phi_2(g)\tilde\chi'(z^{-1})|\omega_{\tilde Y}|(z)|\omega_G|(g)$$

$$\left(\text{since } \int_{\tilde Y}\Phi_1(z^{-1})|\omega_{\tilde Y}|(z) = \int_Y\int_{G_\xi}\Phi_1(hy)|\omega_\xi|(h)|\omega_Y|(y) \text{ by definition}\right)$$

$$= \int_G\int_{\tilde Y}\nu(zg)\Phi_1(g)\Phi_2(zg)\tilde\chi'^{-1}(z)|\omega_{\tilde Y}|(z)|\omega_G|(g) =$$

$$= \int_G \nu(g)\Phi_1(g)\left(\int_{\hat Y}\nu(y)\phi_2(yg)\tilde\chi'^{-1}(y)|\omega_{\hat Y}|(y)\right)|\omega_G|(g) =$$

$$= \int_{G_\xi\backslash G}\left(\int_{\hat Y}\phi_2(yg)\nu\tilde\chi'^{-1}(y)|\omega_{\hat Y}|(y)\right)\phi_1(g)|\omega_X|(g).$$

To prove the statement on eigencharacters recall that, if we denote by $\mathfrak{c}_1$ the eigencharacter of ω_Y then $\tilde\chi'|\mathfrak{c}_1| = \tilde\chi\nu\delta^{-\frac12}$ from the definition of $S^Y_{\tilde\chi^{-1}\nu^{-1}}$. Given the factorization $\omega_{\tilde Y} = \omega_U \wedge o_U^*(\omega_{\hat Y})$ and the fact that ω_U has a $\mathbf{B}$-eigencharacter equal to $\mathfrak{d}_B$, it follows that $\mathfrak{c}_1^{-1}$ = the $\mathbf{B}$-eigencharacter of $\omega_{\tilde Y}$ is equal to $\mathfrak{d}_B\mathfrak{c}$. □

3.5. **Type N geometry.** Before we proceed, we must discuss the case of a type N reflection where additional complications arise and we will need to define other intertwining operators. Let $\mathbf{Y}$ be a Borel orbit of maximal rank. Based on Proposition 3.3.6 we have a distinguished B_0-orbit y_0B_0 on Y (more precisely, on $\mathbf{Y}(\mathfrak{o})$), depending, of course, on the B_0-orbit of the distinguished point $x_0 \in \mathring{\mathbf{X}}(\mathfrak{o})$. Recall that this allows us to identify the geometric quotient $\mathbf{Y}/\mathbf{U}$ with the quotient $\mathbf{A}_Y$ of $\mathbf{A}$, in a unique way up to translations by A_0. We have: $\mathbf{A}_Y = \mathbf{A}/\mathbf{A}_y$, where $\mathbf{A}_y$ is the stabilizer in $\mathbf{B}$, modulo $\mathbf{U}$, of any point $y\in\mathbf{Y}$.

Let α be a simple root such that (Y,α) is of type N. We consider the quotient map: $\mathbf{YP}_\alpha \to \mathbf{YP}_\alpha/\mathcal{R}(\mathbf{P}_\alpha) \simeq \mathcal{N}(\mathbf{T})\backslash\mathbf{PGL}_2$ and let $\mathbf{B}_2$ denote the corresponding Borel subgroup of $\mathbf{PGL}_2$. (As usual, the above isomorphisms are unique up to integral automorphisms.)

The space $\mathbf{X}_2 := N(\mathbf{T})\backslash\mathbf{PGL}_2$ is the space of non-degenerate quadratic forms modulo homotheties. The stabilizer in $\mathbf{B}_2$ (the Borel subgroup of $\mathbf{PGL}_2$) of a point on the open $\mathbf{B}_2$-orbit is $\{\pm I\}$. Therefore, the open B_2-orbits are naturally a torsor for $H^1(k,\pm\{I\})\simeq k^\times/(k^\times)^2$. The same group parametrizes the isomorphism classes of tori over k, and it is easy to see that each open B_2-orbit belongs to a different PGL_2-orbit, with all of the possible isomorphism classes of tori appearing as the connected components of stabilizers in the different PGL_2-orbits.

The quotient map: $\mathbf{YP}_\alpha \to \mathbf{YP}_\alpha/\mathcal{R}(\mathbf{P}_\alpha)\simeq\mathcal{N}(\mathbf{T})\backslash\mathbf{PGL}_2$ induces a map from the set of open B-orbits on $(\mathbf{YP}_\alpha)(k)$ to the set of open B_2-orbits on X_2. The latter comes with a distinguished point (the image of y_0B) and the "kernel" of the composite map: $Y/U = A_Y \to \{B_2\text{-orbits on } X_2\}$ will be denoted by $A_{Y,\alpha}$. The word kernel is justified by the fact that the latter set is a trivialized torsor for $H^1(k,\pm\{I\})$ and the last map is a group homomorphism; in particular, $A_{Y,\alpha}$

is a subgroup of A_Y. Notice that this definition is specific to the root α and this dependence will be a serious obstacle in obtaining a useful explicit formula in those cases.

Let ζ be a coset of $A_{Y,\alpha}$ in A_Y. The points of Y mapping to ζ are those belonging to the preimage of a single open B_2-orbit on X_2; their set will be denoted by Y_ζ. Since the connected component of the isotropy subgroup in the PGL_2-orbit of that B_2-orbit is a torus, we can attach certain invariants to the coset ζ. More precisely, we let $D(\zeta)$ be the discriminant of the splitting field of that torus. It is an element of $\mathfrak{o} \smallsetminus \mathfrak{p}^2$, well-defined up to the action of $(o^\times)^2$. According as $D(\zeta) \in \mathfrak{o}^\times$ or not we will say that "ζ corresponds to an integral, resp. non-integral torus". This is equivalent to ζ cointaining (resp. not containing) $\mathfrak{o}$-points. If ζ corresponds to an integral torus, we will say that it "corresponds to a split, resp. non-split, torus" according as $D(\zeta) \in (o^\times)^2$ or not.

3.6. **Spaces of morphisms and their bases.** Let $\underline{S}^Y_\chi$ denote the space of morphisms $C_c^\infty(X) \to I(\chi)$ spanned by the operators $S^Y_{\tilde\chi}$ with $\tilde\chi$ extending χ. Likewise, let $\underline{\Delta}^Y_\chi$ be the span of the $\Delta^Y_{\tilde\chi}$. The standard bases we will be using for these spaces are the bases of $S^Y_{\tilde\chi}$ and $\Delta^Y_{\tilde\chi}$. (And when we will be expressing linear maps between such spaces as matrices using this basis, unless otherwise stated.)

Unfortunately, in certain cases it is necessary to use a different basis: Let (Y,α) be of type N. We defined above a subgroup $A_{Y,\alpha}$ of A_Y. Now we define the following basis of $\underline{S}^Y_\chi$: It will be indexed by the set of data $(\tilde\chi,\zeta)$ where $\tilde\chi$ runs over representatives for equivalence classes of extensions[5] of χ to A_Y, two of them considered equivalent if they have the same restriction to $A_{Y,\alpha}$; and ζ is a coset of $A_{Y,\alpha}$ in A_Y. Let Y_ζ denote the set of points on Y which map to ζ under $Y \to Y/U \simeq A_Y$. The corresponding operator $S^{\mathring{X}}_{\tilde\chi^{-1}\nu^{-1},\zeta}$ will be given by the formula:

$$\mathrm{ev}_1 \circ S^{Y,\zeta}_{\tilde\chi}(\phi) = \int_{Y_\zeta} \phi(y)\tilde\chi'^{-1}(y)|\omega_Y|(y). \tag{3.8}$$

In other words, the integral is the same as for $S^Y_{\tilde\chi}$, except that integration is restricted to the subset Y_ζ. The adjoint of $S^Y_{\tilde\chi^{-1}\nu^{-1},\zeta}$ will be denoted by $\Delta^Y_{\tilde\chi,\zeta}$.

3.7. **Composing intertwining operators.** A main result of [Sa08] was:

3.7.1. **Theorem.** *For an orbit* $\mathbf{Y}$ *of maximal rank and for almost all* $\chi \in Adm_Y$ *the space* $\underline{S}^Y_\chi$ *is mapped under composition with* T_w *into the space* $\underline{S}^{{}^wY}_{{}^w\chi}$. *More precisely for* $w = w_\alpha$ *a simple reflection, the composition is given as follows:*

- *If* (Y,α) *is of type* G *then* $T_w \underline{S}^Y_\chi = 0$.
- *If* (Y,α) *is of type* U *or* (U,ψ) *or* T *then* $T_w S^Y_{\tilde\chi} \sim S^{{}^wY}_{{}^w\tilde\chi}$.
- *If* (Y,α) *is of type* N *then* $T_w S^Y_{\tilde\chi,\zeta} \sim S^Y_{{}^w\tilde\chi,\zeta}$ *where* $\tilde\chi,\zeta$ *are as at the end of the previous paragraph.*

Here the symbol $\sim$ *denotes equality up to a non-zero rational function of* χ.

By passing to adjoints, it immediately follows that:

[5] Recall that the phrase "extensions to A_Y" is by abuse of language, since $\tilde\chi$ is not a character of A_Y, cf. §3.2.

3.7.2. **Corollary.** *For an orbit* $\mathbf{Y}$ *of maximal rank and for almost all* χ *the space* $\underline{\Delta}^Y_\chi$ *is mapped under composition with* T_w *into the space* $\underline{\Delta}^{^wY}_{^w\chi}$. *More precisely for* $w = w_\alpha$ *a simple reflection, the composition is given as follows:*

- *If* (Y, α) *is of type* G *then* $T_w \underline{\Delta}^Y_\chi = 0$.
- *If* (Y, α) *is of type* U *or* (U, ψ) *or* T *then* $T_w \Delta^Y_{\tilde{\chi}} \sim \Delta^{^wY}_{^w\tilde{\chi}}$.
- *If* (Y, α) *is of type* N *then* $T_w \Delta^Y_{\tilde{\chi},\zeta} \sim \Delta^Y_{^w\tilde{\chi},\zeta}$ *where* $\tilde{\chi}, \zeta$ *are as in* §3.6.

(Notice that $T_{w^{-1}}$ is, up to a rational multiple, adjoint to T_w.)

In particular, for $\mathbf{Y} = \mathring{\mathbf{X}}$ (where we omit the superscript ^{Y} from the notation) we get that $T_w \Delta_{\tilde{\chi}}$ is not identically zero if and only if $w \in [W/W_{P(X)}]$ (coset representatives of minimal length), that $T_w \underline{\Delta}_\chi = \underline{\Delta}_\chi$ if and only if $w \in W_X$ (for generic χ) and that for almost all points in the support of $C_c^\infty(X)^K$ as an $\mathcal{H}(G, K)$-module, the space of eigenvectors is spanned by the K-invariant vectors in the images of $\underline{\Delta}_\chi$ (for suitable χ), since for almost all points all other intertwining operators can be obtained by composing elements of $\underline{\Delta}_\chi$ by some T_w. (Knop's action is transitive on the set of orbits of maximal rank.) Therefore, the goal of computing eigenvectors of the unramified Hecke algebra now becomes:

> *Let* $\phi_{K,\chi}$ *denote a* K*-invariant vector in* $I(\chi)$. *Compute its image under* $\Delta_{\tilde{\chi}}$, *for every* $\tilde{\chi}$.

A priori, this will provide us with all the eigenvectors for almost every point in the support. In §8 we will examine, among other things, to what extent this gives a complete answer for every point.

3.8. **Schwartz space and Fourier transforms.** We now explain a precise choice of intertwining operators which will make our computations easier and more transparent. In fact, we change the setting slightly and replace unramified principal series by functions on $U\backslash G$. We recall that there exists a notion of Schwartz space and Fourier transform for this space [BK98]:

If $\mathbf{G} = \mathbf{SL}_2$ then it is simply the space of compactly supported, locally constant functions on $k^2 \supset k^2 \smallsetminus \{0\} \simeq U\backslash G$ with the equivariant Fourier transform. (The Fourier transform obtained by identifying k^2 with its dual via a non-degenate symplectic form.) Since our spaces have models over $\mathfrak{o}$, we can impose some canonical conditions on our Fourier transforms, namely: The measure on k^2 is such that $\mathfrak{o}^2$ has measure 1 (this is the correct choice only under our simplifying assumption that k is unramified over $\mathbb{Q}_p$); we use a symplectic form which in some basis for $\mathfrak{o}^2$ has the form: $\omega((x, y), (v, w)) = xw - yv$; and we use an additive character ψ on k whose conductor is the ring of integers $\mathfrak{o}$. (We choose and fix such a character.) Using this data, Fourier transform on k^2 is given by the formula:

$$(Ff)(v, w) = \int_{k^2} f(x, y)\psi(-xw + yv)dxdy. \tag{3.9}$$

In general, one defines for every simple reflection w_α in the Weyl group an equivariant Fourier transform: $\mathcal{F}_{w_\alpha} : L^2(U\backslash G) \to L^2(U\backslash G)$ as follows: Let $\mathbf{P}_\alpha$ be the parabolic of semisimple rank one corresponding to the simple root α. Then we have $[\mathbf{P}_\alpha, \mathbf{P}_\alpha]/\mathbf{U}_{P_\alpha} \simeq \mathbf{SL}_2$ and $\mathbf{U}\backslash[\mathbf{P}_\alpha, \mathbf{P}_\alpha] \simeq \mathbb{A}^2 \smallsetminus \{0\}$. Notice that the integral structure on $\mathbf{G}$ defines an integral structure on $\mathbb{A}^2 \smallsetminus \{0\}$, therefore after fixing the former isomorphism the latter is determined up to multiplication by an element of $\mathfrak{o}^\times$. Then one defines $\mathcal{F}_{w_\alpha}$ to be the unique equivariant transform which is equal to the $\mathbf{SL}_2$-equivariant Fourier transform on the restriction of a continuous function

in $L^2(U\backslash G)$ to $U\backslash[P_\alpha, P_\alpha]$. It is known that the equivariant Fourier transforms compose as follows: $\mathcal{F}_w \circ \mathcal{F}_{w'} = \mathcal{F}_{ww'}$, hence they define a unitary action of the Weyl group on $L^2(U\backslash G)$. Braverman and Kazhdan define the Schwartz space to be the smallest subspace which contains $C_c^\infty(U\backslash G)$ and is closed under all Fourier transforms. The Schwartz space comes with a canonical K-invariant vector c_0, which is stable under all Fourier transforms and whose restriction to the $[L_\alpha, L_\alpha]$-orbit of $U \cdot 1$, for each simple root α, is equal, under the above identifications, to the characteristic function of $\mathfrak{o}^2$ in k^2.

We define the action of A on $L^2(U\backslash G)$ as follows, so that it is *unitary*:

$$(L_a f)(Ug) = \delta^{-\frac{1}{2}}(a) f(Uag).$$

This way, we have $\mathcal{F}_w(L_a f) = L_{wa}\mathcal{F}_w f$.

We can extend Fourier transforms uniquely to *tempered* distributions (i.e. distributions on the Schwartz space) and tempered generalized funtions. For almost every χ, the elements of the principal series $I(\chi)$ are tempered, and by the equivariance properties we have $F_w : I(\chi) \to I({}^w\tilde{\chi})$.

Equivalently, consider the surjective map: $\mathcal{S}(U\backslash G) \to I(\chi)$ given by the rational continuation of the integral:

$$f \mapsto \int_A L_a f \ \chi^{-1}(a) da.$$

(We normalize $\mathrm{Vol}(A_0) = 1$.) The Fourier transforms and the intertwining operators between the $I(\chi)'s$ fit into the commutative diagram:

$$\begin{array}{ccc} \mathcal{S}(U\backslash G) & \xrightarrow{\int} & I(\chi) \\ {\scriptstyle \mathcal{F}_w}\big\downarrow & & \big\downarrow{\scriptstyle T_w} \\ \mathcal{S}(U\backslash G) & \xrightarrow{\int} & I({}^w\chi) \end{array}$$

for T_w suitably normalized. The operators T_w normalized in this way are the operators F_w. Notice that the F_w's satisfy: $F_w^* = F_{w^{-1}}$ (recall that the smooth dual of $I(\chi)$ is $I(\chi^{-1})$).

The reason for working with Schwartz space and Fourier transforms, instead of principal series and their intertwining operators, is that computations will be much easier there; in particular, we will be able to avoid all the mysterious cancellations and simplifications that one discovers a posteriori in all other variants of the Casselman-Shalika method.

3.9. **An Iwahori-invariant vector of small support.** The starting point for (all variants of) the Casselman-Shalika method is the observation that it is easy to compute the value of the functional $\Delta^Y_{\tilde{\chi}}$ on translates of an Iwahori-invariant vector of small support. More precisely, let $\Phi_J \in \mathcal{S}(U\backslash G)$ denote the characteristic function of $U \cdot J$.

Lemma. *For $x \in A^+_X$ we have:* $\Delta^Y_{\tilde{\chi}}(\Phi_J)(x) = \begin{cases} 0 & \text{if } \mathbf{Y} \neq \mathring{\mathbf{X}} \\ \mathrm{Vol}(J) \cdot \tilde{\chi}\delta^{-\frac{1}{2}}(x) & \text{otherwise.} \end{cases}$

Proof.

$$\Delta^Y_{\tilde{\chi}}(\Phi_J)(x) = \frac{\left\langle \Delta^Y_{\tilde{\chi}}(\Phi_J), 1_{xJ}|\omega_X|\right\rangle_X}{|\omega_X|(xJ)} = \frac{\left\langle \Phi_J, S^Y_{\tilde{\chi}^{-1}\nu^{-1}}(1_{xJ}|\omega_X|)\right\rangle_{U\backslash G}}{|\omega_X|(xJ)} =$$

$$= \frac{|\omega_{U\backslash G}|(U\backslash UJ)}{|\omega_X|(xJ)} \cdot \nu(x) \cdot \mathrm{ev}_{U1} \circ S^Y_{\tilde{\chi}^{-1}\nu^{-1}}(1_{xJ}).$$

By Axiom 2.4.2 we have: $xJ \subset xB_0$. Therefore, from the definition of $S^Y_{\tilde{\chi}^{-1}\nu^{-1}}$ (3.2) we get: $\mathrm{ev}_{U1} \circ S^Y_{\tilde{\chi}^{-1}\nu^{-1}}(1_{xJ}) = 0$ unless $\mathbf{Y} = \mathring{\mathbf{X}}$, in which case:

$$\nu(x)\,\mathrm{ev}_{U1} \circ S^Y_{\tilde{\chi}^{-1}\nu^{-1}}(1_{xJ}) = |\omega_X|(xJ)\tilde{\chi}\delta^{-\frac{1}{2}}(x).$$

□

3.10. **Functional equations.** Using Fourier transforms, and the fact that $F^*_{w^{-1}} = F_w$, Corollary 3.7.2 can be restated as follows:

For every $w \in W$ there is a rational family of linear operators: $\underline{b}^Y_w(\chi) : \underline{\Delta}^Y_\chi \to \underline{\Delta}^{{}^wY}_{{}^w\chi}$ such that $F_w\Delta = \underline{b}^Y_w(\chi)\Delta$ for $\Delta \in \underline{\Delta}^Y_\chi$. Moreover, for $Y = \mathring{X}$ the operator $\underline{b}_w(\chi)$ is non-zero if and only if $w \in [W/W_{P(X)}]$, and for w a simple reflection we have explicit bases which make the matrix of b^Y_w diagonal. Since $F_{w_1} \circ F_{w_2} = F_{w_1w_2}$, the $\underline{b}^Y_w$ satisfy the cocycle relations:

$$\underline{b}^Y_{w_1w_2}(\chi) = \underline{b}^{{}^{w_2}Y}_{w_1}({}^{w_2}\chi)\underline{b}^Y_{w_2}(\chi). \tag{3.10}$$

As a matter of notation, *we will be identifying $\underline{b}^Y_w(\chi)$ with its matrix in the bases* $[\Delta^Y_{\tilde{\chi}}]_{\tilde{\chi}}, [\Delta^{{}^wY}_{{}^w\tilde{\chi}}]_{\tilde{\chi}}$.

Suppose that we knew the matrices $\underline{b}_w(\chi)$ (recall that when we ignore the orbit as a superscript we mean $\mathring{X}$). Then it would be easy to compute values of $\Delta_{\tilde{\chi}}$ applied to every vector of the form $\mathcal{F}_{w^{-1}}\Phi_J$ as follows:

$$[\Delta_{\tilde{\chi}}(\mathcal{F}_{w^{-1}}\Phi_J)(x)]_{\tilde{\chi}} = [F_w\Delta_{\tilde{\chi}}(\Phi_J)(x)]_{\tilde{\chi}} = \underline{b}_w(\chi) \cdot [\Delta^{{}^w\mathring{X}}_{{}^w\tilde{\chi}}(\Phi_J)(x)]_{\tilde{\chi}}.$$

This vanishes unless ${}^w\mathring{\mathbf{X}} = \mathring{\mathbf{X}}$, i.e. unless $w \in W_X$, in which case from the previous lemma we get:

$$[\Delta_{\tilde{\chi}}(\mathcal{F}_{w^{-1}}\Phi_J)(x)]_{\tilde{\chi}} = \underline{b}_w(\chi) \cdot [\mathrm{Vol}(J) \cdot {}^w\tilde{\chi}\delta^{-\frac{1}{2}}(x)]_{\tilde{\chi}} \tag{3.11}$$

Therefore, the computation of the Hecke eigenvectors will be straightforward if we can perform the following two steps:

(1) Compute the functional equations $F_w[\Delta^Y_{\tilde{\chi}}]_{\tilde{\chi}} = \underline{b}^Y_w(\chi)[\Delta^{{}^wY}_{{}^w\tilde{\chi}}]_{\tilde{\chi}}$.
(2) Express a non-zero K-invariant Schwartz function as a linear combination of the functions $\mathcal{F}_{w^{-1}}\Phi_J$, $w \in W$.

Remark. The second step is independent of the particular case that we are examining (i.e. independent of H, Ψ), and it suffices to do it once and for all. It would therefore be possible to retrieve it, for example, from the work of Casselman on zonal spherical functions. However, we construct a K-invariant vector in an independent way, which in the sequel will help us avoid all the mysterious cancellations which are usually found in the literature and will lead us directly to simple formulae. Notice also that the requirement that a linear combination of the $\mathcal{F}_{w^{-1}}\Phi_J$'s be non-zero is automatically satisfied for any non-trivial linear combination. This follows from (3.11) (applied to any choice of $\mathbf{H}$) and the linear independence of the characters ${}^w\chi$, for χ in general position.

4. Construction of a K-invariant vector

4.1. **The case of SL_2.** In this section we perform the second step of §3.10.

We start by considering the case of SL_2. Then $\Phi_J = 1_{\mathfrak{p}\times\mathfrak{o}^\times} = 1_{\mathfrak{p}\times\mathfrak{o}} - 1_{\mathfrak{p}\times\mathfrak{p}}$. Since $1_{\mathfrak{p}\times\mathfrak{p}}$ is K-invariant we see that $\Phi_J \equiv 1_{\mathfrak{p}\times\mathfrak{o}}$ up to K-invariants and the equivariant Fourier transform of Φ_J is, up to K-invariants, equal to

$$\mathcal{F}1_{\mathfrak{p}\times\mathfrak{o}} \equiv q^{-1}1_{\mathfrak{o}\times\mathfrak{p}^{-1}} = L_{\begin{pmatrix} \varpi^{-1} & \\ & \varpi\end{pmatrix}} 1_{\mathfrak{p}\times\mathfrak{o}}.$$

Hence

$$L_{\begin{pmatrix} \varpi^{-1} & \\ & \varpi\end{pmatrix}}\Phi_J - \mathcal{F}\Phi_J \equiv L_{\begin{pmatrix} \varpi^{-1} & \\ & \varpi\end{pmatrix}}1_{\mathfrak{p}\times\mathfrak{o}} - \mathcal{F}1_{\mathfrak{p}\times\mathfrak{o}} \equiv 0$$

up to K-invariants; equivalently, the vector:

$$\Phi_J - L_{\begin{pmatrix} \varpi & \\ & \varpi^{-1}\end{pmatrix}}\mathcal{F}\Phi_J \tag{4.1}$$

is K-invariant.

4.2. **The general case.** We now return to an arbitrary G. To simplify notation, for each co-root $\check\alpha$ and each $\Phi \in \mathcal{S}(U\backslash G)$ we will denote $L_{e^{\check\alpha}(\varpi)}\Phi$ by $e^{\check\alpha}\Phi$. This is consistent with the fact that the image of $L_{e^{\check\alpha}(\varpi)}\Phi$ in $I(\chi)$ under the natural morphism (integration with respect to the A-action against the character χ^{-1}) is $e^{\check\alpha}(\chi) := \chi(e^{\check\alpha}(\varpi))$ times the image of Φ.

For each simple root α let $K_\alpha \subset K$ denote the inverse image, through the reduction map, of $\mathbf{P}_\alpha(\mathbb{F}_q)$. Evidently, K is generated by all the K_α since they contain the Iwahori and representatives for the simple reflections in the Weyl group. Using the SL_2 case that we examined above, it is now easy to show:

4.2.1. **Lemma.** *The vector* $\Phi_K := \mathrm{Vol}(J)^{-1}\sum_{w\in W}\prod_{\alpha>0,w^{-1}\alpha<0}(-1)e^{\check\alpha}\mathcal{F}_w\Phi_J$ *is K-invariant.*

Proof. It suffices to show that for any simple positive root β it is K_β-invariant.

For the minimal parabolic P_β we denote by W_β the Weyl group of its Levi, and by $[W/W_\beta]$ the canonical set of representatives of W/W_β cosets consisting of representatives of minimal length. Notice that if $w = w'w_\beta$, with $w' \in [W/W_\beta]$, then $\{\alpha|\alpha>0, w^{-1}\alpha<0\} = \{\alpha|\alpha>0, w'^{-1}\alpha<0\}\cup\{w'\beta\}$. Hence:

$$\mathrm{Vol}(J)\Phi_K = \sum_{w\in[W/W_\beta]}\prod_{\alpha>0,w^{-1}\alpha<0}(-1)e^{\check\alpha}\left(\mathcal{F}_w\Phi_J - e^{w\check\beta}\mathcal{F}_{ww_\beta}\Phi_J\right) =$$
$$= \sum_{w\in[W/W_\beta]}\prod_{\alpha>0,w^{-1}\alpha<0}(-1)e^{\check\alpha}\mathcal{F}_w\left(\Phi_J - e^{\check\beta}\Phi_J\right)$$

and the latter is K_β-invariant. ∎

Notice that we can write: $\prod_{\alpha>0,w^{-1}\alpha<0}e^{\check\alpha}(\chi) = e^{\check\rho - w\check\rho}(\chi)$. Combining this with (3.11) we get:

4.2.2. **Theorem.** *For almost every χ, a basis of eigenvectors of $\mathcal{H}(G,K)$ on $C^\infty(X)$ with eigencharacter corresponding to χ is given by the formula:*

$$[\Omega_{\tilde\chi}(x)]_{\tilde\chi} = [\Delta_{\tilde\chi}(\Phi_K)(x)]_{\tilde\chi} = e^{\check\rho}(\chi)\delta^{-\frac12}(x)\sum_{w\in W_X}\sigma(w)e^{-\check\rho}({}^w\chi)\underline{b}_w(\chi)\cdot[{}^w\tilde\chi(x)]_{\tilde\chi} \tag{4.2}$$

for $x \in A_X^+$, where $\sigma(w)$ is the sign of w as an element of W and the matrices $\underline{b}_w(\chi)$ are given by the functional equations:

$$F_w[\Delta_{\tilde{\chi}}]_{\tilde{\chi}} = \underline{b}_w(\chi)[\Delta_{{}^w\tilde{\chi}}]_{\tilde{\chi}}. \tag{4.3}$$

Remarks. (1) There is some abuse of notation here (which we will consistently practice!), in that the expression ${}^w\tilde{\chi}\delta^{-\frac{1}{2}}(x)$ should be taken as a whole and not as a product of two factors, since ${}^w\tilde{\chi}$ is not, in general, a character of A_X.

(2) In the case that (Y,α) is never of type N (for Y of maximal rank and α a simple root) then we actually know from Corollary 3.7.2 that the matrices $\underline{b}_w^Y(\chi)$ are diagonal. If in addition $\tilde{\chi}$ is unramified we may write: $\tilde{\chi}(x_{\check{\lambda}}) = e^{-\check{\lambda}}(\tilde{\chi})$ where $x_{\check{\lambda}} = e^{-\check{\lambda}}(\varpi) \in A_X^+$ for any uniformizer ϖ, and the formula becomes more pleasant:

$$\Omega_{\tilde{\chi}}(x_{\check{\lambda}}) = e^{\check{\rho}}(\chi)e^{-\check{\lambda}}(\delta^{\frac{1}{2}}) \sum_{w\in W_X} \sigma(w) b_w(\tilde{\chi}) e^{-\check{\rho}+\check{\lambda}}({}^w\tilde{\chi}) \tag{4.4}$$

where the $b_w(\tilde{\chi})$ are now scalars given by the functional equations: $F_w\Delta_{\tilde{\chi}} = b_w(\tilde{\chi})\Delta_{{}^w\tilde{\chi}}$.

The following will be useful later:

4.2.3. **Lemma.** *We have $\Omega_{\delta^{\frac{1}{2}}}(x) = 1$ for every x.*

Proof. By our assumption that $\mathbf{X}$ carries an eigen-volume form, and by the form of the morphisms $S_{\tilde{\chi}}$, it follows that $S_{\nu^{-1}\delta^{-\frac{1}{2}}}$ furnishes a morphism: $C_c^\infty(X) \to \nu^{-1}$ or, tensoring by ν, a morphism: $M_c^\infty(X) \to 1$ (the trivial representation). Therefore its dual, as a morphism: $I(\delta^{\frac{1}{2}}) \to C^\infty(X)$, factors through the trivial representation. Moreover, the image of Φ_K via the morphism $\mathcal{S}(U\backslash G) \to I(\chi)$ is finite at $\delta^{\frac{1}{2}}$ (see (4.5) below). Therefore, $\Omega_{\delta^{\frac{1}{2}}}$ is a constant function, and by its form (4.2) it has to be equal to 1. ∎

Remark. The lemma does not apply to the case of $C^\infty(X, \mathcal{L}_\Psi)$, with non-trivial Ψ, since there is no G-eigenmeasure valued in $\mathcal{L}_\Psi$ in that case.

4.3. **Normalization.** It is good to keep track of how our eigenfunctions are normalized, so we collect here the properties of $\Omega_{\tilde{\chi}} = \Delta_{\tilde{\chi}}(\Phi_K)$:

1. Iwahori normalization: Let $\phi_{J,\chi} \in I(\chi)$ be the vector with $\phi_{J,\chi}|_K = 1_J$. Then $\phi_{J,\chi}$ is the image of Φ_J and $\Delta_{\tilde{\chi}}(\phi_{J,\chi}) = \mathrm{Vol}(J) = \frac{1}{(K:J)}$.
2. "Spherical vector" normalization: Let $\phi_{K,\chi} \in I(\chi)$ be the unramified vector with $\phi_{K,\chi}(bk) = \chi\delta^{\frac{1}{2}}(b)$ for every $b \in B$. To determine the value of $\Delta_{\tilde{\chi}}(\phi_{K,\chi})$ we must determine its relation with the image of Φ_K. Let $\int_\chi$ denote the natural morphism: $\mathcal{S}(U\backslash G) \to I(\chi)$. We claim:

$$\phi_{K,\chi} = Q^{-1} \prod_{\check{\alpha}>0} \frac{1-q^{-1}e^{\check{\alpha}}}{1-e^{\check{\alpha}}}(\chi) \int_\chi \Phi_K \tag{4.5}$$

where $Q = \frac{\mathrm{Vol}(K)}{\mathrm{Vol}(Jw_lJ)} = \prod_{\check{\alpha}>0} \frac{1-q^{-1}e^{\check{\alpha}}}{1-e^{\check{\alpha}}}(\delta^{\frac{1}{2}})$. To prove this we need a result which we will prove later, together with a result of Casselman and Shalika. Let Δ_χ refer to the morphism corresponding to the Whittaker model, i.e.

$\mathbf{H} = \mathbf{U}^-$, the opposite unipotent subgroup, with a generic character Ψ. We prove later that:

$$\Delta_\chi(\Phi_K)(e^{\check\lambda}) = e^{-\check\lambda}(\delta^{\frac{1}{2}})e^{\check\rho}(\chi)\sum_W \sigma(w)e^{-\check\rho+\check\lambda}({}^w\chi).$$

The corresponding formula of [CS80, Theorem 5.4] is:

$$\Delta_\chi^{CS}(\phi_{K,\chi})(e^{\check\lambda}) = \prod_{\check\alpha>0}\frac{1-q^{-1}e^{\check\alpha}}{1-e^{\check\alpha}}(\chi)e^{-\check\lambda}(\delta^{\frac{1}{2}})e^{\check\rho}(\chi)\sum_W \sigma(w)e^{-\check\rho+\check\lambda}({}^w\chi)$$

where Δ_χ^{CS} is normalized so that $\Delta_\chi^{CS}(\phi_{J^-,\chi}) = 1$, where $\phi_{J^-,\chi}$ denotes the element in $I(\chi)$ with $\phi_{J^-,\chi}|_K = 1_{B_0J^-}$, where J^- is the Iwahori subgroup corresponding to the opposite Borel. Comparing the two normalizations, since clearly in this case $\Delta_\chi(\phi_{J^-,\chi}) = (U_0 : U_1)\Delta_\chi(\phi_J)$ (here U_1 is the kernel in U_0 of the reduction map to $\mathbf{U}(\mathbb{F}_q)$) and also since $Q = (U_0 : U_1)\operatorname{Vol}(J)$, our claim follows. (Notice that in [CS80] the spherical subgroup has not been put in opposite position from the Borel and the representatives for K-orbits are all dominant. Therefore, one has to substitute $e^{w_l\check\lambda}$ for a in [CS80, Theorem 5.4] – where w_l is the longest element of the Weyl group – to get the correct formula.)

3. "Basic vector" normalization: In their study [BK98] of $\mathcal{S}(U\backslash G)$ Braverman and Kazhdan introduce a "basic vector" $c_0 \in \mathcal{S}(U\backslash G)^K$ which generates $\mathcal{S}(U\backslash G)$ over $\mathcal{H}(A, A_0) \otimes \mathcal{H}(G, K)$. It is related to 1_{UK} as follows:

$$1_{UK} = \prod_{\check\alpha>0}(1-q^{-1}e^{\check\alpha})c_0$$

where $e^{\check\alpha}$ denotes the corresponding element of $\mathcal{H}(A, A_0)$ (acting via the normalized left A-action on $U\backslash G$) (see [BK98, eq. (3.15), (3.22)]). Therefore $\int_\chi c_0 = \prod_{\check\alpha>0}(1-q^{-1}e^{\check\alpha})(\chi)\phi_K$ and from our previous computation:

$$c_0 = \frac{Q^{-1}\Phi_0}{\prod_{\check\alpha>0}(1-e^{\check\alpha})}. \tag{4.6}$$

Notice that, indeed, using the expression of Lemma 4.2.1, one immediately verifies that c_0 is invariant under Fourier transforms.

At this point we can also discuss the relationship between the intertwining operators F_w and the "classical" intertwining operators T_w defined as the rational continuation of the integral:

$$T_w(\phi) = \int_{U\cap w^{-1}Uw\backslash U}\phi(\tilde w u)du$$

(where $\tilde w \in \mathcal{N}(\mathbf{A})(\mathfrak{o})$ is a representative for w). By [Ca80, Theorem 3.1] one has:

$$T_w\phi_{K,\chi} = \prod_{\check\alpha>0, w\check\alpha<0}\frac{1-q^{-1}e^{\check\alpha}}{1-e^{\check\alpha}}(\chi)\phi_{K,{}^w\chi}. \tag{4.7}$$

On the other hand, we have

$$F_w(\phi_{K,\chi}) = \prod_{\check\alpha>0, w\check\alpha<0}\frac{1-q^{-1}e^{\check\alpha}}{1-q^{-1}e^{-\check\alpha}}(\chi)\phi_{K,{}^w\chi}$$

and therefore:

$$F_w = \prod_{\check\alpha>0, w\check\alpha<0} \frac{1-e^{\check\alpha}}{1-q^{-1}e^{-\check\alpha}}(\chi)T_w \tag{4.8}$$

5. Functional equations

5.1. **Reduction to rank one.** In this section we compute the proportionality factors in the functional equations:

$$F_w[\Delta^Y_{\tilde\chi}]_{\tilde\chi} = \underline{b}^Y_w(\chi)[\Delta^{^wY}_{^w\tilde\chi}]_{\tilde\chi}$$

or equivalently:

$$F_w[S^Y_{\tilde\chi^{-1}\nu^{-1}}]_{\tilde\chi} = \underline{b}^Y_w(\chi)[S^{^wY}_{^w\tilde\chi^{-1}\nu^{-1}}]_{\tilde\chi}$$

when $w=w_\alpha$ is a simple reflection and $\mathbf{Y}$ is an orbit of maximal rank. For general w, the functional equations follow from the cocycle relations (3.10). Assume $(\mathbf{Y},\alpha)$ not of type G. We have already exhibited basis elements of $\underline{\Delta}^Y_\chi, \underline{\Delta}^{^wY}_{^w\chi}$ which map to a multiple of each other under Fourier transform. As a matter of notation, the (scalar) quotient of $F_w\Delta^Y_{\tilde\chi}$ by $\Delta^{^wY}_{^w\tilde\chi}$, in cases U, (U,ψ) and T will be denoted by $b^Y_w(\tilde\chi)$, and the quotient of $F_w\Delta^{Y,\zeta}_{\tilde\chi}$ by $\Delta^{^wY,\zeta}_{^w\tilde\chi}$ in case N will be denoted by $b^{Y,\zeta}_w(\tilde\chi)$. Hence, it is enough to compute these scalar quotients for all simple reflections $w=w_\alpha$.

Remark. Though our notation below is adapted to the cases U and T, the same discussion holds with obvious modifications for cases N and (U,ψ).

To reduce to the case of $\mathbf{SL}_2$, we proceed as follows:

We can compose $S^Y_{\tilde\chi^{-1}\nu^{-1}}$ with the "restriction" morphism to get a sequence of P_α-morphisms:

$$C_c^\infty(X) \xrightarrow{S^Y_{\tilde\chi^{-1}\nu^{-1}}} C^\infty_{\text{temp}}(U\backslash G) \xrightarrow{\text{Res}} C^\infty_{\text{temp}}(U\backslash P_\alpha).$$

We temporarily denote by $S'^Y_{\tilde\chi^{-1}\nu^{-1}}$ the composition. (The dependence on α has been suppressed from the notation.)

We have a commutative diagram of P_α-morphisms:

$$\begin{array}{ccc} C^\infty_{\text{temp}}(U\backslash G) & \xrightarrow{\text{Res}} & C^\infty_{\text{temp}}(U\backslash P_\alpha) \\ \mathcal{F}_{w_\alpha}\downarrow & & \downarrow\mathcal{F}_{w_\alpha} \\ C^\infty_{\text{temp}}(U\backslash G) & \xrightarrow{\text{Res}} & C^\infty_{\text{temp}}(U\backslash P_\alpha). \end{array} \tag{5.1}$$

Therefore the morphisms $S'^Y_{\tilde\chi^{-1}\nu^{-1}}$ satisfy *the same functional equations with respect to* w_α as the morphisms $S^Y_{\tilde\chi^{-1}\nu^{-1}}$.

Clearly, from the definition of $S^Y_{\tilde\chi^{-1}\nu^{-1}}$, the morphism $S'^Y_{\tilde\chi^{-1}\nu^{-1}}$ factors through restriction of elements of $C_c^\infty(X)$ to $\overline{YP_\alpha}$. To compute the scalar proportionality factors, it is enough to restrict our P_α-morphisms to the P_α-stable subspace of $C_c^\infty(X)$ of those elements whose restriction to $\overline{YP_\alpha}$ is supported in YP_α. For those, we can factorize the morphism as:

$$S'^Y_{\tilde\chi^{-1}\nu^{-1}}\otimes 1: C_c^\infty(YP_\alpha)\otimes\nu \to C_c^\infty((\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha)(k), \delta_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}})\otimes\nu \to$$
$$\to \text{Ind}^{L_\alpha}_{B_\alpha}(\chi^{-1}\nu^{-1}\delta^{\frac12})\otimes\nu \simeq \text{Ind}^{L_\alpha}_{B_\alpha}(\chi^{-1}\delta^{\frac12}) \subset C^\infty_{\text{temp}}(U_\alpha\backslash L_\alpha)$$

according to the decomposition of eigenforms as in §3.3. Here the notation is as follows: ξ is a point on YP_α (to be chosen more carefully later), and $\mathbf{H}_\alpha$ denotes the image of $\mathbf{G}_\xi \cap \mathbf{P}_\alpha$ in $\mathbf{L}_\alpha = \mathbf{P}_\alpha/\mathbf{U}_{P_\alpha}$. The first arrow is "integration along U_{P_α}-orbits".

We denote the second arrow in the above sequence by $S^{Y,\alpha}_{\tilde\chi^{-1}\nu^{-1}}$, fix invariant measures[6] on $U_\alpha\backslash L_\alpha$ and $(\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha)(k)$ (this admits an invariant measure by inspection of the quasi-affine $\mathbf{PGL}_2$-spherical varieties), and denote by $\Delta^{Y,\alpha}_{\tilde\chi}$ the adjoint:

$$\Delta^{Y,\alpha}_{\tilde\chi} : \mathcal{S}(U_\alpha\backslash L_\alpha) \to C^\infty((\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha)(k), \delta^{-1}_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}}) \otimes \nu^{-1} \xrightarrow{\sim}$$
$$\xrightarrow{\sim} C^\infty((\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha)(k), \delta^{-1}_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}}\nu^{-1}).$$

Notice that this factors through the dual of $\mathrm{Ind}^{L_\alpha}_{B_\alpha}(\chi^{-1}\delta^{\frac12})$, which is canonically identified with $\mathrm{Ind}^{L_\alpha}_{B_\alpha}(\chi e^\alpha\delta^{-\frac12})$. The L_α-morphisms $\Delta^{Y,\alpha}_{\tilde\chi}$ *satisfy the same functional equations as the morphisms* $\Delta^Y_{\tilde\chi}$. Let $\hat{\mathbf{Y}}_\alpha := \mathbf{U}_\alpha\backslash\mathbf{B}_\alpha\mathbf{H}_\alpha \subset \mathbf{U}_\alpha\backslash\mathbf{L}_\alpha$. In analogy with Lemma 3.4.2, we have the following result (notice that the line bundle $\mathcal{L}_{\delta^{-1}_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}}\nu^{-1}}$ over $\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha(k)$ comes with a canonical trivialization of its pull-back to L_α, therefore it makes sense to "evaluate at 1"):

5.1.1. **Lemma.** *For $\phi \in C^\infty_c(U_\alpha\backslash L_\alpha)$ and χ in a suitable region of convergence we have:*

$$\mathrm{ev}_1 \circ \Delta^{Y,\alpha}_{\tilde\chi}(\phi) = \int_{\hat Y_\alpha} \phi(y)\nu\tilde\chi'^{-1}(y)|\omega_{\hat Y_\alpha}|(y)$$

for a suitable $\mathbf{A}\times\mathbf{H}_\alpha$-eigenform $\omega_{\hat Y_\alpha}$ on $\hat{\mathbf{Y}}_\alpha$ with eigencharacter:

$$\mathfrak{c}_\alpha \times \mathfrak{d}_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}}\mathfrak{n}$$

(unnormalized action of $\mathbf{A}$) where $\mathfrak{c}_\alpha$ and $\tilde\chi'$ satisfy: $\nu^{-1}\tilde\chi'|\mathfrak{c}_\alpha|^{-1} = e^\alpha\delta^{-\frac12}\tilde\chi$.

For $(\mathbf{Y},\alpha)$ of type N the same holds for $\mathrm{ev}_1\circ\Delta^{Y,\alpha}_{\tilde\chi,\zeta}$, provided that $\xi\in Y_\zeta$.

Proof. The proof is completely analogous to that of Lemma 3.4.2. The only thing to notice is that in case N, if we consider the image of $\tilde Y_\alpha = (\mathbf{B}_\alpha\mathbf{H}_\alpha)(k)$ in Y under the action map $g\mapsto \xi\cdot g^{-1}$, it belongs to $Y_{\alpha,\zeta}$. □

For the computation, we pick the point ξ as follows: In cases U, (U,ψ) and T we choose $\xi\in\mathbf{Y}(\mathfrak{o})$, and more precisely we choose ξ in the distinguished B_0-orbit (cf. Proposition 3.3.6). The choice of ξ in case N will be discussed in §5.8. (We caution the reader that in case N it will not be possible, in general, to choose ξ in the distinguished B_0-orbit on Y (cf. §5.8), and for that reason the function $\nu\tilde\chi'^{-1}(y)$ of Lemma 5.1.1 on $\hat Y_\alpha$ will not, in general, be equal to one at $U_\alpha 1$.) To simplify notation, we temporarily denote the distribution $\mathrm{ev}_1\circ\Delta^{Y,\alpha}_{\tilde\chi}$ on $U_\alpha\backslash L_\alpha$ (respectively $\mathrm{ev}_1\circ\Delta^{Y,\alpha}_{\tilde\chi,\zeta}$ in case N) by $\Delta'_{\tilde\chi}$ (resp. $\Delta'_{\tilde\chi,\zeta}$). Hence:

$$\Delta'_{\tilde\chi} : \mathcal{S}(U_\alpha\backslash L_\alpha)\to\mathbb{C}$$

is an $A\times H_\alpha$-eigendistribution, with eigencharacter (considering the unnormalized action of A):

$$\chi^{-1}e^{-\alpha}\delta^{\frac12}\times\delta_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}}\nu.$$

To complete the reduction to $\mathbf{SL}_2$, it suffices to consider the factorization: $\mathbf{L}_\alpha = \mathbf{Z}_\alpha\mathbf{L}'_\alpha$ (where $\mathbf{Z}_\alpha$ is the centre of $\mathbf{L}_\alpha$) and to notice that the distribution $\Delta_{\tilde\chi}$ is

[6]The measures have to be chosen in compatible ways, as we did in §3.4.

smooth along Z_α-orbits with respect to Haar measure on Z_α. More precisely, we can pick a small subgroup-neighborhood N of $1 \in Z_\alpha$ such that:

- A neighborhood of $U_\alpha\backslash L'_\alpha$ is isomorphic to $U_\alpha\backslash L'_\alpha \times N$ under the action map.
- The restriction of $\Delta'_{\tilde\chi}$ to this neighborhood can be written as: Haar measure on $N \times$ a distribution $\Delta^2_{\tilde\chi}$ on $U_\alpha\backslash L'_\alpha$.

Since the kernel of Fourier transform is supported on $U_\alpha\backslash L'_\alpha \times U_\alpha\backslash L'_\alpha$, it is enough for the functional equations to compute the Fourier transform of $\Delta^2_{\tilde\chi}$ (having fixed a Haar measure on N which is independent of χ). We are free to pick the Haar measure on N (and hence vary $\Delta^2_{\tilde\chi}$ up to a constant which is independent of $\tilde\chi$), but we need to make the same choices when relating the distributions $\Delta^2_{\tilde\chi}$ coming from two orbits $\mathbf{Y}, \mathbf{Z}$ of maximal rank in the same $\mathbf{P}_\alpha$-orbit. In particular, the corresponding distributions $\Delta^2_{\tilde\chi}$ will be related to each other in the way discussed in §3.3.4.

5.2. **The result.** The computation has now been reduced to the case of $\mathbf{SL}_2$. To present the functional equations, we need to introduce some extra language: If $\check\lambda : \mathbb{G}_m \to \mathbf{A}_Y$ is a cocharacter, its image is an algebraic subgroup $\mathbf{M}_{\check\lambda}$ of $\mathbf{A}_Y$ we say that a character $\tilde\chi$ of A_Y is $\check\lambda$-*unramified* if it is unramified on $M_{\check\lambda}$.

Finally, in the split case T we introduce some extra data coming from the geometry of the spherical variety. More precisely, $\mathbf{YP}_\alpha$ is the union of $\mathbf{Y}$ and two k-rational divisors $\mathbf{D}, \mathbf{D}'$. They define valuations $\check v_D, \check v_{D'} : k(Y) \to \mathbb{Z}$ which, restricted to $k(Y)^{(B)}$ define homomorphisms: $\mathcal{X}(Y) \to \mathbb{Z}$. These, in turn, can be identified with coweights $\check v_D, \check v_{D'}$ into $\mathbf{A}_Y$. It is known that $\check v_D = -^{w_\alpha}\check v_{D'}$ and that $\check v_D + \check v_{D'} = \check\alpha$ as elements of $\mathcal{X}(Y)^*$. (As we will see later, the image of $\check v_D$ in $\mathbf{A}_Y$ coincides with the image the stabilizer in $\mathbf{B}$ modulo $\mathbf{U}$ of a point on $\mathbf{D}$.)

We are ready now to state the functional equations for $\Delta^Y_{\tilde\chi}$ (resp. $\Delta^Y_{\tilde\chi,\zeta}$) and for $w = w_\alpha$, a simple reflection. The data that we need is: The type of $(\mathbf{Y}, \alpha)$ (U, (U,ψ), T split, T non-split or N); in case T split, the coweights $\check v_D, \check v_{D'}$ (as we saw, knowledge of one implies knowledge of the other) and the modular character $\delta_{(U_{P_\alpha})_\xi}$, as a character of the stabilizer of ξ in P_α; in case N, whether the coset ζ corresponds to an integral/non-integral, split/non-split torus.

Finally, we need to know the character $\tilde\chi$. We remind that $\tilde\chi$ is not really a character of A_Y but rather a character of its preimage in $\mathbf{A}(\bar k)$ which satisfies the condition:

$$\tilde\chi\nu\delta^{\frac12}|_{B_\xi} = \delta_{B_\xi}.$$

In any case, for a co-root $\check\alpha$ the quantity $e^{\check\alpha}(\tilde\chi) = e^{\check\alpha}(\chi)$ makes sense. It also makes sense in case T to consider the expression $e^{\check v_D}(\tilde\chi\nu\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi})$, as we will explain in §5.6.

5.2.1. **Theorem.** *In cases U, (U,ψ) and T the functional equations read as follows:*

Case U: *We have*

$$\mathcal{F}_{w_\alpha}\Delta^Y_{\tilde\chi} = \Delta^{w_\alpha Y}_{w_\alpha\tilde\chi} \cdot \begin{cases} -e^{-\check\alpha}\frac{1-q^{-1}e^{-\check\alpha}}{1-e^{-\check\alpha}}(\chi) \text{ if } \alpha \text{ lowers } Y \\ \\ -e^{-\check\alpha}\frac{1-e^{\check\alpha}}{1-q^{-1}e^{\check\alpha}}(\chi) \text{ if } \alpha \text{ raises } Y. \end{cases}$$

Case U,ψ: *We have*

$$\mathcal{F}_{w_\alpha}\Delta^Y_{\tilde\chi} = \Delta^Y_{w_\alpha\tilde\chi}.$$

Case T, split: *We have*

$$\mathcal{F}_{w_\alpha}\Delta^Y_{\tilde\chi} = \Delta^Y_{w_\alpha\tilde\chi}\cdot\frac{(1-q^{-1}e^{-\check v_D})(1-q^{-1}e^{-\check v_{D'}})}{(1-e^{\check v_D})(1-e^{\check v_{D'}})}(\tilde\chi\nu\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi})\Delta^{w_\alpha}\chi$$

if $\tilde\chi$ is $\check v_D$-unramified, and

$$\mathcal{F}_{w_\alpha}\Delta^Y_{\tilde\chi} = \Delta^Y_{w_\alpha\tilde\chi}\cdot e^{-m\check\alpha}(\chi)$$

if it is $\check v_D$-ramified of conductor $\mathfrak{p}^m$.

Case T, non-split: *We have*

$$\mathcal{F}_{w_\alpha}\Delta^Y_{\tilde\chi} = \Delta^Y_{w_\alpha\tilde\chi}\cdot\frac{1-q^{-1}e^{-\check\alpha}(\chi)}{1-q^{-1}e^{\check\alpha}(\chi)}$$

if $\tilde\chi$ is $\frac{\check\alpha}{2}$-unramified, and

$$\mathcal{F}_{w_\alpha}\Delta^Y_{\tilde\chi} = \Delta^Y_{w_\alpha\tilde\chi}\cdot e^{-\check\alpha}(\chi)$$

otherwise.

Case N: *Here we have four cases, according as the coset ζ corresponds to a split or non-split, integral or non-integral torus, and in each of these cases we have to distinguish between $\tilde\chi$ being $\frac{\check\alpha}{2}$-ramified or not. The functional equations for this case are given in §5.8.*

5.3. **Preliminaries of the computation.** We are going to compute the proportionality constant case-by-case. We will use many times the following fact: Consider usual Fourier transform of functions and distributions on k, with respect to our fixed character ψ and the measure dx normalized as: $dx(\mathfrak{o})=1$. The Fourier transform of a function f on k will be denoted by $\hat f$. By Tate's thesis, the Fourier transform of the distribution $\chi(x)d^\times x$ (where $dx^\times$ denotes some fixed multiplicative Haar measure on $k^\times$) is equal to:

$$\frac{1-q^{-1}\chi^{-1}(\varpi)}{1-\chi(\varpi)}\chi^{-1}(x)|x|d^\times x = \frac{1-q^{s-1}}{1-q^{-s}}|x|^{1-s}d^\times x$$

if $\chi(x)=|x|^s$ is unramified, and:

$$q^{-m}\tau(\chi)\cdot\chi^{-1}(x)|x|d^\times x$$

if χ is a ramified character of conductor $\mathfrak{p}^m$, where $\tau(\chi)$ is the Gauss sum:

$$\tau(\chi) = \sum_{\epsilon\in\mathfrak{o}^\times/(1+\mathfrak{p}^m)}\chi(\varpi^{-m}\epsilon)\psi(\varpi^{-m}\epsilon).$$

Indeed, we know a priori from equivariance properties that the Fourier transform of $\chi(x)d^\times x$ must be a multiple of $\chi^{-1}(x)|x|d^\times x$, and that multiple can be computed from the formula: $\langle f,g\rangle = \left\langle \hat f,\hat g\right\rangle$. Tate computes: $\langle 1_\mathfrak{o},\chi d^\times x\rangle =: \zeta(1_\mathfrak{o},\chi|\bullet|) = (1-\chi(\varpi))^{-1}$ if χ is unramified. If $\chi=\eta\cdot|\bullet|^s$ is ramified, with η a unitary character, we compute the zeta integral of $f_m := \psi(x)1_{\mathfrak{p}^{-m}}$ whose Fourier transform is $\hat f_m = q^m 1_{1+\mathfrak{p}^m}$ and we get: $\langle f_m,\chi d^\times x\rangle = \zeta(f_m,\eta|\bullet|^s)) = q^{ms}\tau(\eta)\operatorname{Vol}(\mathfrak{p}^m)$ and $\left\langle \hat f_m,\chi^{-1}|\bullet|d^\times x\right\rangle = \zeta(\hat f_m,\eta^{-1}|\bullet|^{-s+1}) = q^m\operatorname{Vol}(\mathfrak{p}^m)$.

Notice that here we are using our simplifying assumption that k is unramified over $\mathbb{Q}_p$; these equations would have to be modified otherwise.

5.4. **Case U.** Let us identify $\mathbf{L}'_\alpha$ with $\mathbf{SL}_2$ and $\mathbf{U}_\alpha\backslash\mathbf{L}'_\alpha$ with $\mathbb{A}^2$ over $\mathfrak{o}$. The distribution $\Delta^2_{\tilde\chi}$ (defined at the end of §5.1) is here an $\mathbf{A}_2\times\mathbf{U}_\alpha$-eigendistribution. Based on Lemma 5.1.1, depending on whether Y is raised or lowered by α, the distribution $\Delta^2_{\tilde\chi}$ on $U_\alpha\backslash L'_\alpha$ is, up to an integral change of coordinates and up to a common scalar multiple one of the following two:

$$\Delta_{1,\chi}=|x|^{s_1}dxdy \text{ and } \Delta_{2,\chi}=|y|^{s_2}\delta_0(x)dy$$

where $s_1,s_2\in\mathbb{C}$ is such that $q^{s_1+1}=q^{s_2}=e^{-\check\alpha}(\chi\nu)=e^{-\check\alpha}(\chi)$.

Now we see immediately that

$$F_{w_\alpha}\Delta_{1,\chi}=\frac{1-q^{s_1}}{1-q^{-s_1-1}}\Delta_{2,{}^{w_\alpha}\chi}=\frac{1-q^{-1}e^{-\check\alpha}(\chi)}{1-e^{\check\alpha}(\chi)}\Delta_{2,{}^{w_\alpha}\chi}$$

and

$$F_{w_\alpha}\Delta_{2,\chi}=\frac{1-q^{s_2}}{1-q^{-s_2-1}}\Delta_{1,{}^{w_\alpha}\chi}=\frac{1-e^{-\check\alpha}(\chi)}{1-q^{-1}e^{\check\alpha}(\chi)}\Delta_{1,{}^{w_\alpha}\chi}$$

Hence, for (α,Y) of type U, we have:

$$\boxed{b^Y_{w_\alpha}(\tilde\chi)=\begin{cases}-e^{-\check\alpha}\frac{1-q^{-1}e^{-\check\alpha}}{1-e^{-\check\alpha}}(\chi) \text{ if } \alpha \text{ lowers } Y\\ \\ -e^{-\check\alpha}\frac{1-e^{\check\alpha}}{1-q^{-1}e^{\check\alpha}}(\chi) \text{ if } \alpha \text{ raises } Y.\end{cases}} \tag{5.2}$$

5.4.1. *Example.* Let $\mathbf{G}'=\mathbf{G}\times\mathbf{G}$, $\mathbf{H}=\mathbf{G}^{\mathrm{diag}}$, and consider the spherical variety $\mathbf{X}=\mathbf{G}=\mathbf{H}\backslash\mathbf{G}'$ of $\mathbf{G}'$ with action $(g_1,g_2)\cdot x=g_1^{-1}xg_2$. Let us choose a Borel subgroup $\mathbf{B}'=\mathbf{B}^-\times\mathbf{B}$, so that $\mathbf{HB}'$ is open. The little Weyl group is generated by $w_\alpha w_{\tilde\alpha}$, where $-\alpha$ is a simple root of $\mathbf{B}^-$ in the first copy of $\mathbf{G}$ and $\tilde\alpha$ the corresponding root of $\mathbf{B}$ in the second copy. The admissible characters χ' are of the form: $\chi^{-1}\otimes\chi$. (We put χ^{-1} in the first copy so that the eigenfunction $\Omega_{\chi^{-1}\otimes\chi}$ is proportional to the matrix coefficient $\langle\pi(g)v,\tilde v\rangle$ with $\pi=I(\chi)$ and $v,\tilde v$ the unramified vectors.) Notice that $w_{\tilde\alpha}$ lowers the open orbit to some orbit $\mathbf{Y}_\alpha$, which w_α then raises back to the open orbit. Hence we compute: $e^{-\check\alpha}e^{\check{\tilde\alpha}}\cdot b_{w_\alpha w_{\tilde\alpha}}(\chi')=\frac{1-q^{-1}e^{-\check\alpha}}{1-e^{-\check\alpha}}(\chi)\frac{1-e^{-\check\alpha}}{1-q^{-1}e^{-\check\alpha}}(\chi^{-1})$ and the formula reads:

$$\Omega_\chi(g_{\check\lambda})=e^{-\check\lambda}(\delta^{\frac12})\sum_W\prod_{\check\alpha>0.w\check\alpha<0}\frac{1-q^{-1}e^{-\check\alpha}}{1-e^{-\check\alpha}}\frac{1-e^{\check\alpha}}{1-q^{-1}e^{\check\alpha}}(\chi)e^{\check\lambda}({}^w\chi)$$

which up to a factor which is independent of $\check\lambda$ is equal to:

$$e^{-\check\lambda}(\delta^{\frac12})\sum_{w\in W}\prod_{\check\alpha>0}\frac{1-q^{-1}e^{\check\alpha}}{1-e^{\check\alpha}}e^{\check\lambda}({}^w\chi).$$

This is, of course, Macdonald's formula for zonal spherical functions which was reproven by Casselman in [Ca80, Theorem 4.2].[7]

5.5. **Case U,ψ:** Here the distribution $\Delta^2_{\tilde\chi}$ has eigencharacter ψ under the action of U. Under the same identifications as above we have, up to an integral change of coordinates:

$$\Delta^2_{\tilde\chi}=|x|^s\psi^{-1}\left(\frac{y}{x}\right)dxdy$$

where $s\in\mathbb{C}$ is such that $q^{s+1}=e^{-\check\alpha}(\chi)$.

[7] To compare with the result in *loc.cit.*, recall that our $\check\lambda$ is anti-dominant.

Here we see that

$$F_{w_\alpha}|x|^s\psi^{-1}\left(\frac{y}{x}\right)dxdy = |x|^{-s-2}\psi^{-1}\left(\frac{y}{x}\right)dxdy$$

in other words:

$$F_{w_\alpha}\Delta_\chi^2 = \Delta_{w_\alpha\chi}^2$$

and hence:

$$\boxed{b_{w_\alpha}^Y(\chi) = 1} \tag{5.3}$$

5.5.1. *Example.* Let $H = U$ with the standard Whittaker character. Then $W_X = W$, all simple roots are of type (U,ψ), and we have:

$$\Omega_\chi(g_{\check\lambda}) = e^{-\check\lambda}(\delta^{\frac{1}{2}})e^{\check\rho}(\chi)\sum_W \sigma(w)e^{-\check\rho+\check\lambda}({}^w\chi).$$

This[7] is the Shintani-Casselman-Shalika formula [CS80].

5.5.2. *Example.* Let $G = \mathrm{GL}_{2n}$ and let H be a conjugate of the Shalika subgroup such that HB is open, equipped with the Shalika character Ψ. The Shalika subgroup of is the semidirect product of GL_n, embedded diagonally in the $\mathrm{GL}_n\times\mathrm{GL}_n$ parabolic, with the unipotent radical of this parabolic; the Shalika character is the complex character $(g,u)\mapsto\psi(\mathrm{tr}\,u)$ of this subgroup. In this case, if we enumerate $w_1,\dots,w_{2n-1}$ the simple reflections in the Weyl group of G, the little Weyl group is generated by the elements w_iw_{2n-i} $(1\le i\le n-1)$ and w_n, hence can be identified with the Weyl group of the subgroup $\mathrm{Sp}_{2n}(\mathbb{C})$ of the dual group $\check G = \mathrm{GL}_{2n}(\mathbb{C})$. The character χ is of the form $\chi_0\otimes{}^{w_0}\chi_0^{-1}$, where χ_0 is a character of the maximal torus of GL_n, and w_0 denotes the longest Weyl element of GL_n. The action of the element w_iw_{2n-1} on Knop's graph is by lowering $\mathring{\mathbf{X}}$ to some orbit $\mathbf{Y}_i$ and then raising $\mathbf{Y}_i$ back to the open orbit, so the corresponding factor is: $e^{\check\alpha}e^{\check{\tilde\alpha}}(\chi)b_{w_\alpha w_{\tilde\alpha}}(\chi) = \frac{1-q^{-1}e^{-\check\alpha}}{1-e^{-\check\alpha}}\frac{1-e^{\check\alpha}}{1-q^{-1}e^{\check\alpha}}(\chi)$ (where $\alpha,\tilde\alpha$ are the roots corresponding to w_i, w_{2n-i}, and we have used the fact that χ is of the aforementioned form). On the other hand, the pair $(\mathring{X},w_n)$ is of type (U,ψ), hence the corresponding factor is $e^{\check\alpha}(\chi)b_\alpha(\chi) = \frac{1-e^{\check\alpha}}{1-e^{-\check\alpha}}(\chi)$. Notice that χ can be considered as an element of the maximal torus of $\mathrm{Sp}_{2n}(\mathbb{C})\subset\check G$, and the coroots $\check\alpha$ which appear in the above factors are the roots of $\mathrm{Sp}_{2n}(\mathbb{C})$. Hence the unramified Shalika function is given by:

$$\Omega_\chi(g_{\check\lambda}) = e^{-\check\lambda}(\delta^{\frac{1}{2}})\sum_{W_X}\prod_{\alpha\in\Phi^S_{\mathrm{Sp}_{2n}},\alpha>0,w\alpha<0}\frac{1-q^{-1}e^{-\check\alpha}}{1-e^{-\check\alpha}}\frac{1-e^{\check\alpha}}{1-q^{-1}e^{\check\alpha}}(\chi)$$

$$\cdot\prod_{\alpha\in\Phi^L_{\mathrm{Sp}_{2n}},\alpha>0,w\alpha<0}\frac{1-e^{\check\alpha}}{1-e^{-\check\alpha}}(\chi)e^{\check\lambda}(\chi)$$

which, up to a factor independent of $\check\lambda$, is:

$$e^{-\check\lambda}(\delta^{\frac{1}{2}})\sum_{W_X}\sigma(w)\prod_{\alpha\in\Phi^S_{\mathrm{Sp}_{2n}}}(1-q^{-1}e^{\check\alpha})e^{-\check\rho+\check\lambda}({}^w\chi)$$

($\sigma(w)$ denoting the sign of w as an element of W), the formula of [Sa06, Theorem 2.1].[8] The symbols $\Phi^S_{\mathrm{Sp}_{2n}}$ and $\Phi^L_{\mathrm{Sp}_{2n}}$ denote the sets of short and long roots of Sp_{2n}, respectively.

5.6. **Case T, split.** The goal of the discussion that follows is to identify the distribution $\Delta^2_{\tilde\chi}$ in case T, split. The following is an easy fact:

5.6.1. **Lemma.** *In case T, the group $\mathbf{A}\cap\mathbf{H}_\alpha$ is central in $\mathbf{L}_\alpha$.*

Proof. Its image in $\mathbf{L}_\alpha/\mathcal{R}(\mathbf{L}_\alpha)\simeq\mathbf{PGL}_2$ is trivial. □

This implies that the antidiagonal copy: $g\mapsto(g^{-1},g)$ of this group is the kernel of the morphism: $\mathbf{A}\times\mathbf{H}_\alpha\to\mathrm{Aut}(\mathbf{U}_\alpha\backslash\mathbf{L}_\alpha)$. Denote the quotient by $\overline{\mathbf{A}\times\mathbf{H}_\alpha}$, i.e.:

$$\overline{\mathbf{A}\times\mathbf{H}_\alpha}:=(\mathbf{A}\times\mathbf{H}_\alpha)/(\mathbf{A}\cap\mathbf{H}_\alpha)^{\mathrm{adiag}}\hookrightarrow\mathrm{Aut}(\mathbf{U}_\alpha\backslash\mathbf{L}_\alpha).$$

Let $\check v_D,\check v_{D'}:\mathbb{G}_m\to\mathbf{A}_Y$ denote the cocharacters defined in §5.2. We will prove:

5.6.2. **Proposition.** (1) *There is a natural way to identify $\check v_D,\check v_{D'}$ with cocharacters $\widetilde{\check v_D},\widetilde{\check v_{D'}}$ into $\overline{\mathbf{A}\times\mathbf{H}_\alpha}$ such that:*
   i) *Their compositions with the map $\overline{\mathbf{A}\times\mathbf{H}_\alpha}\to\mathbf{A}_Y$ are equal to $\check v_D,\check v_{D'}$, respectively.*
   ii) *Their images are automorphisms which preserve the subvariety $\mathbf{U}_\alpha\backslash\mathbf{L}'_\alpha$.*
   iii) *Under an $\mathfrak{o}$-isomorphism: $\mathbf{U}_\alpha\backslash\mathbf{L}'_\alpha\simeq\mathbb{A}^2_{x,y}$ and up to an integral change of coordinates, the automorphisms $\check v_D(a),\check v_{D'}(a)$ restrict to $\mathbf{U}_\alpha\backslash\mathbf{L}'_\alpha$ as: $(x,y)\mapsto(ax,y)$ and $(x,y)\mapsto(x,ay)$.*

(2) *The distribution $\Delta^2_{\tilde\chi}$ is an eigendistribution for $\widetilde{\check v_D}(k^\times),\widetilde{\check v_{D'}}(k^\times)$, with eigencharacter equal to the restriction of $\tilde\chi^{-1}e^{-\alpha}\delta^{\frac12}\otimes\delta_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}}\nu$ to their image.*

Notice that the statement about the eigencharacter makes sense, because we have $\tilde\chi^{-1}e^{-\alpha}\delta^{\frac12}=\delta_{G_\xi\cap U_{P_\alpha}\backslash U_{P_\alpha}}\nu$ on $A\cap H_\alpha$. Indeed, the right hand side is equal to $\delta_{U_{P_\alpha}}\delta^{-1}_{(U_{P_\alpha})_\xi}$ and in this case we have: $(U_{P_\alpha})_\xi=U_\xi$. Therefore, the equality follows from (3.7). By a slight abuse of notation, we will be denoting the pull-back of this character to $k^\times$ by $\tilde\chi^{-1}\nu^{-1}\delta^{-\frac12}\delta_{(U_{P_\alpha})_\xi}\circ e^{\check v_D}$ (and similarly for D').

Proof. Let $z\in\mathbf{D}$ be any point. Consider the series of quotients: $\mathbf{B}\to\mathbf{A}\to\mathbf{A}_Y$. Let $\mathbf{B}'\subset\mathbf{B}$, $\mathbf{A}'\subset\mathbf{A}$ be the preimages of $\check v_D(\mathbb{G}_m)$. We claim:

The subgroup $\mathbf{B}'$ coincides with the stabilizer of the $\mathbf{U}$-orbit of z.

An element $b\in\mathbf{B}$ is in the stabilizer of $z\mathbf{U}$ if and only if z and $z\cdot b$ cannot be separated by a regular $\mathbf{B}$-eigenfunction on $\mathbf{D}$. The pair $(\mathbf{Y},\alpha)$ being of type T, every $\mathbf{B}$-eigenfunction on $\mathbf{D}$ extends to a $\mathbf{B}$-eigenfunction on $\mathbf{YP}_\alpha$. By definiton of $\check v_D$, the eigencharacters of eigenfunctions which restrict non-trivially to $\mathbf{D}$ are precisely those which are orthogonal to $\check v_D$. This proves the claim.

Under the identification: $\mathbf{YP}_\alpha\simeq(\mathbf{P}_\alpha)_\xi\backslash\mathbf{P}_\alpha$ (following from a choice of point ξ), we may let $\hat{\mathbf{D}}$ denote the orbit of $\mathbf{A}\times\mathbf{H}_\alpha$ on $\mathbf{U}\backslash\mathbf{P}_\alpha=\mathbf{U}_\alpha\backslash\mathbf{L}_\alpha$ corresponding to the $\mathbf{B}$-orbit $\mathbf{D}$ on $\mathbf{YP}_\alpha$. Then the above claim translates to the following:

For every $\hat z\in\hat{\mathbf{D}}$ the stabilizer of z in $\mathbf{A}\times\mathbf{H}_\alpha$ is isomorphic to $\mathbf{A}'$.

[8] Again, here, one needs to substitute $\check\lambda$ by $w_{\check\lambda}\check\lambda$ to arrive at the formula of *loc.cit.* Notice that here $e^{-\check\lambda}=e^{w_l\check\lambda}$ where w_l is the longest element of the Weyl group, so the signs of the coweights inside of the W_X-sum will be inverted.

Clearly, this isomorphism does not depend on the choice of z. Since the antidiagonal copy of $\mathbf{A}\cap\mathbf{H}$ acts trivially, we get:

For every $\hat{z}\in\hat{\mathbf{D}}$ *the stabilizer of* z *in* $\overline{\mathbf{A}\times\mathbf{H}_\alpha}$ *is canonically isomorphic to* $\check{v}_D(\mathbb{G}_m)$.

Hence the cocharacter $\widetilde{\check{v}_D}$ of the proposition.

We have a morphism: $\mathbf{U}_\alpha\backslash\mathbf{L}_\alpha\to\mathbf{L}'_\alpha\backslash\mathbf{L}_\alpha$. Therefore the stabilizer, in $\mathbf{A}\times\mathbf{L}_\alpha$, of any point on $\mathbf{U}_\alpha\backslash\mathbf{L}'_\alpha$ stabilizes $\mathbf{U}_\alpha\backslash\mathbf{L}'_\alpha$, as well. Hence claim (i).

Identify $\mathbf{U}_\alpha\backslash\mathbf{L}'_\alpha\simeq\mathbb{A}^2_{x,y}\smallsetminus\{0\}$. For a suitable choice of $\mathfrak{o}$-isomorphisms, we can identify the non-open orbits of $\mathrm{stab}_{\overline{\mathbf{A}\times\mathbf{H}_\alpha}}(\mathbf{U}_\alpha\backslash\mathbf{L}'_\alpha)$ with the two coordinate axes. Let f be a $\mathbf{B}$-eigenfunction on $\mathbf{H}_\alpha\backslash\mathbf{L}_\alpha$ with eigencharacter e^α. Equivalently, $f\circ\iota$ (where ι denotes inversion in the group) can be thought of as an $\mathbf{A}\times\mathbf{H}_\alpha$-eigenfunction on $\mathbf{U}_\alpha\backslash\mathbf{L}_\alpha$ with eigencharacter $e^\alpha\times 1$. From the definitions, $f\circ\widetilde{\check{v}_D}(a)=af$. Since $\widetilde{\check{v}_D}(a)$ stabilizes the points of a coordinate axis and preserves the other axis, it follows that $\widetilde{\check{v}_D}(a)$ is the automorphism: $(x,y)\mapsto(ax,y)$ or $(x,y)\mapsto(y,ax)$.

The statement about the eigencharacter follows from the eigencharacter of $\Delta^{Y,\alpha}_{\tilde{\chi}}$, cf. Lemma 5.1.1. □

Therefore, up to an integral change of variables we have on $\mathbb{A}^2_{x,y}$:

$$\Delta^2_{\tilde{\chi}}=\eta_D(x)\eta_{D'}(y)d^\times x d^\times y$$

where $\eta_D=\tilde{\chi}\nu\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi}\circ\check{v}_D$, and similarly for $\eta_{D'}$. Its Fourier transform is, therefore, as follows:

(1) If $\tilde{\chi}$ is $\check{v}_D$–unramified (equivalently, since $\check{v}_D+\check{v}_{D'}=\check{\alpha}$, if it is $\check{v}_{D'}$-unramified):

$$F_{w_\alpha}\Delta^2_{\tilde{\chi}}=\frac{(1-q^{-1}e^{-\check{v}_D})(1-q^{-1}e^{-\check{v}_{D'}})}{(1-e^{\check{v}_D})(1-e^{\check{v}_{D'}})}(\tilde{\chi}\nu\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi})\Delta^2_{w_\alpha\tilde{\chi}}$$

Consequently:

$$\boxed{b^Y_{w_\alpha}(\tilde{\chi})=\frac{(1-q^{-1}e^{-\check{v}_D})(1-q^{-1}e^{-\check{v}_{D'}})}{(1-e^{\check{v}_D})(1-e^{\check{v}_{D'}})}(\tilde{\chi}\nu\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi}).}\tag{5.4}$$

Remark. The vanishing of the terms $(1-e^{\check{v}_D})(\tilde{\chi}\nu\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi})$ and $(1-e^{\check{v}_{D'}})(\tilde{\chi}\nu\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi})$ in the denominator is precisely the condition for the orbit of D (resp. D') to support a morphism from $I(\tilde{\chi})$, in other words a condition for $\Delta^{\mathring{D}}_{\tilde{\chi}}$ (resp. $\Delta^{\mathring{D}'}_{\tilde{\chi}}$) to be defined. This, of course, was expected – see [Ga99] or [Sa08, §4.6].

(2) If $\tilde{\chi}$ is $\check{v}_D$-ramified:

$$F_{w_\alpha}\Delta^2_{\tilde{\chi}}=q^{-2m}\tau(\eta_D)\tau(\eta_{D'})\eta_D(-1)\Delta^2_{w_\alpha\tilde{\chi}}.$$

We can simplify the above expression as follows: From the fact that $\hat{\hat{f}}(x)=f(-x)$ and the formula $\widehat{\eta(x)d^\times x}=q^{-m}\tau(\eta)\eta^{-1}(x)|x|d^\times x$ for a ramified character η of conductor $\mathfrak{p}^m$ it follows that $\tau(\eta)\tau(\eta^{-1})=\eta(-1)q^m$. Applying this to $\eta=\eta_D$ and taking into account that $\eta_{D'}=\tilde{\chi}\nu\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi}\circ$

$e^{\check\alpha}\cdot\eta_D^{-1}\Rightarrow\tau(\eta_{D'})=\chi\nu\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi}\circ e^{\check\alpha}(\varpi^{-m})\cdot\tau(\eta^{-1})=q^m e^{-m\check\alpha}(\chi)\tau(\eta^{-1})$
we get:

$$F_{w_\alpha}\Delta^2_{\tilde\chi}=e^{-m\check\alpha}(\chi)\Delta^2_{w_\alpha\tilde\chi}.$$

Hence:

$$\boxed{b^Y_{w_\alpha}(\tilde\chi)=e^{-m\check\alpha}(\chi).} \tag{5.5}$$

5.6.3. *Example.* Let $\mathbf{G}=(\mathbf{PGL}_2)^3$ and $\mathbf{H}$ a $\mathbf{G}(\mathfrak{o})$-conjugate of the diagonal copy of $\mathbf{PGL}_2$ such that $\mathbf{HB}$ is open. For instance, if we take for $\mathbf{B}$ three copies of the group of upper triagonal matrices, then $\mathbf{H}$ can be the diagonal copy of $\mathbf{PGL}_2$ conjugated by the element $\left(\begin{pmatrix}1&\\&1\end{pmatrix},\begin{pmatrix}&1\\-1&\end{pmatrix},\begin{pmatrix}1&\\1&1\end{pmatrix}\right)$. Here the little Weyl group is the whole Weyl group and $A^*_X=A^*$. Let $\alpha_1,\alpha_2,\alpha_3$ denote the simple positive roots in each copy respectively. Then we easily see that $(\mathring{X},\alpha_i)$ is of type T and that there are three orbits D_1,D_2,D_3 of codimension one, with $\check v_{D_i}=\frac12(-\check\alpha_i+\sum_{j\ne i}\check\alpha_j)$ and the $D_j, j\ne i$ being the two orbits in the P_{α_i}-orbit of $\mathring{X}$. Moreover, δ_{B_ξ} is trivial. Therefore:

$$b_{\alpha_i}(\chi)=\frac{(1-q^{-\frac12}e^{\frac{-\check\alpha_i+\check\alpha_j-\check\alpha_k}{2}})(1-q^{-\frac12}e^{\frac{-\check\alpha_i-\check\alpha_j+\check\alpha_k}{2}})}{(1-q^{-\frac12}e^{\frac{\check\alpha_i+\check\alpha_j-\check\alpha_k}{2}})(1-q^{-\frac12}e^{\frac{\check\alpha_i-\check\alpha_j+\check\alpha_k}{2}})}(\chi)$$

and, up to a factor independent of $\check\lambda$ (see also Example 7.2.4), $\Omega_\chi(g_{\check\lambda})$ is equal to:

$$e^{-\check\lambda}(\delta^{\frac12})\sum_W\sigma(w)(1-q^{-\frac12}e^{\frac{\check\alpha_1+\check\alpha_2-\check\alpha_3}{2}})(1-q^{-\frac12}e^{\frac{\check\alpha_1-\check\alpha_2+\check\alpha_3}{2}})\cdot$$
$$\cdot(1-q^{-\frac12}e^{\frac{-\check\alpha_1+\check\alpha_2+\check\alpha_3}{2}})(1-q^{-\frac12}e^{\frac{\check\alpha_1+\check\alpha_2+\check\alpha_3}{2}})e^{-\check\rho+\check\lambda}({}^w\chi).$$

5.7. **Case T non-split:** The analysis here simpler than in the split case, because all one-dimensional non-split tori of $\mathbf{L}_\alpha$ are contained in $\mathbf{L}'_\alpha$ and therefore $\mathbf{H}_\alpha\cap\mathbf{L}'_\alpha$ is a spherical subgroup of $\mathbf{L}'_\alpha$. Under an $\mathfrak{o}$-identification of $\mathbf{U}_\alpha\backslash\mathbf{L}'_\alpha$ with $\mathbb{A}^2_{x,y}\smallsetminus\{0\}$, the group $\mathbf{H}_\alpha\cap\mathbf{L}'_\alpha$ is the special orthogonal group of a non-degenerate, non-split binary quadratic form Q, and therefore:

$$\Delta^2_{\tilde\chi}=\eta(Q(x,y))dxdy$$

where $\eta=\tilde\chi\delta^{-\frac12}\circ e^{\frac{\check\alpha}{2}}$. By our assumption that the action of the group on $\mathbf{X}$ is smooth over $\mathfrak{o}$, up to an integral change of coordinates we have $Q(x,y)=x^2+\kappa y^2$, where $-\kappa$ is a non-square unit element. We can write $\eta=\eta_2\cdot|\bullet|^{\frac{s}{2}}$ where $q^{s+1}=e^{-\check\alpha}(\chi)$ and η_2 is either the trivial character or a ramified quadratic character.

A fact which will make our computations easier is that if $-\kappa$ is not a square then $|x^2+\kappa y^2|=\max\{|x^2|,|\kappa y^2|\}$. Since, moreover, κ is a unit, the value of $\eta_2(x^2+\kappa y^2)$ only depends on the restriction of η_2 on units. We will compute the constant in the functional equations through the Parseval formula:

$$\left\langle\Delta^2_{\tilde\chi},f\right\rangle=\left\langle\mathcal{F}\Delta^2_{\tilde\chi},\mathcal{F}f\right\rangle$$

with $f=\mathcal{F}f=1_{\mathfrak{o}^2}$.

A basic relation which will be used repeatedly is

$$\int_{|x|<|y|\le 1}|y|^s dxdy=\frac{q^{-1}(1-q^{-1})}{1-q^{-s-2}}.$$

$\tilde{\chi}$ *is* $\frac{\check{\alpha}}{2}$*-unramified.* We compute:

$$\left\langle \Delta^2_{\tilde{\chi}}, 1_{\mathfrak{o}^2}\right\rangle = \int_{\mathfrak{o}^2} |x^2+\kappa y^2|^{\frac{s}{2}} dxdy = \int_{|x|<|y|\le 1} |y|^s dxdy + \int_{|y|<|x|\le 1} |x|^s dxdy +$$
$$+ \int_{|x|=|y|\le 1} |x|^s dxdy = 2\frac{q^{-1}(1-q^{-1})}{1-q^{-s-2}} + \frac{(1-q^{-1})^2}{1-q^{-s-2}} = \frac{1-q^{-2}}{1-q^{-s-2}}.$$

Similarly, for $\Delta^2_{w_\alpha \tilde{\chi}} = |x^2+\kappa y^2|^{-\frac{s}{2}-1}$ we will get:

$$\left\langle \Delta^2_{w\tilde{\chi}}, 1_{\mathfrak{o}^2}\right\rangle = \frac{1-q^{-2}}{1-q^s}$$

and the quotient of the two is:

$$b^Y_{w_\alpha}(\tilde{\chi}) = \frac{1-q^s}{1-q^{-s-2}}.$$

Hence:

$$\boxed{b^Y_{w_\alpha}(\tilde{\chi}) = \frac{1-q^{-1}e^{-\check{\alpha}}(\chi)}{1-q^{-1}e^{\check{\alpha}}(\chi)}.} \tag{5.6}$$

$\tilde{\chi}$ *is* $\frac{\check{\alpha}}{2}$*-ramified.* It is an easy exercise in harmonic analysis and Gauss sums to verify the following identity:

$$\int_{\mathfrak{o}^\times} \eta_2(\kappa+x^2) = \begin{cases} 0 & \text{, if } -1 \in (k^\times)^2 \\ -2q^{-1} & \text{, otherwise.} \end{cases}$$

(Recall that $-\kappa$ is not a square.)

Now we compute:

$$\left\langle \Delta^2_{\tilde{\chi}}, 1_{\mathfrak{o}^2}\right\rangle = \int_{\mathfrak{o}^2} \eta_2(x^2+\kappa y^2)|x^2+\kappa y^2|^{\frac{s}{2}} dxdy =$$
$$= \eta_2(\kappa)\int_{|x|<|y|\le 1} |y|^s dxdy + \int_{|y|<|x|\le 1} |x|^s dxdy + \int_{|x|=|y|\le 1} \eta_2(x^2+\kappa y^2)|y|^s dxdy =$$
$$= (1-\eta_2(-1))\frac{q^{-1}(1-q^{-1})}{1-q^{-s-2}} + \int_{\mathfrak{o}} |y|^{s+1} dy \int_{\mathfrak{o}^\times} \eta_2(x^2+\kappa y^2) =$$
$$= (1-\eta_2(-1))\frac{q^{-1}(1-q^{-1})}{1-q^{-s-2}} + \begin{cases} 0 & \text{, if } -1 \in (k^\times)^2 \\ -2q^{-1}\frac{1-q^{-1}}{1-q^{-s-2}} & \text{, otherwise.} \end{cases} = 0.$$

Hence the value of the distribution $\Delta^2_{\tilde{\chi}}$ on a K-invariant function is zero. (There is no contradiction here; this just says that the corresponding eigenfunction $\Omega_{\tilde{\chi}}$ on $C^\infty(\mathbf{T}\backslash \mathbf{SL}_2)$ has value zero at the chosen point – not that it vanishes identically.) We will use instead the function $\Phi_J = 1_{\mathfrak{p}\times\mathfrak{o}^\times}$ whose Fourier transform has been computed in §4.1:

$$\Delta^2_{\tilde{\chi}}(\Phi_J) = \int_{\mathfrak{p}\times\mathfrak{o}^\times} \eta_2(x^2+\kappa y^2)|x^2+\kappa y^2|^{\frac{s}{2}} dxdy = q^{-1}(1-q^{-1})\eta_2(\kappa)$$

and we have $\mathcal{F}(\Phi_J) \equiv L_{\check{\alpha}(\varpi^{-1})}\Phi_J$ modulo K-invariants, whence by the vanishing of $\Delta^2_{\tilde{\chi}}(1_{\mathfrak{o}^2})$:

$$b_{w_\alpha}(\tilde{\chi})q^{-1}(1-q^{-1})\eta_2(\kappa) = b_{w_\alpha}(\tilde{\chi})\left\langle \Delta^2_{w\tilde{\chi}}, \Phi_J\right\rangle = \left\langle \mathcal{F}\Delta^2_{\tilde{\chi}}, \Phi_J\right\rangle =$$
$$= \left\langle \Delta^2_{\tilde{\chi}}, \mathcal{F}\Phi_J\right\rangle = \left\langle \Delta^2_{\tilde{\chi}}, L_{\check{\alpha}(\varpi^{-1})}\Phi_J\right\rangle = e^{-\check{\alpha}}(\chi)q^{-1}(1-q^{-1})\eta_2(\kappa).$$

Therefore:

$$b^Y_{w_\alpha}(\tilde\chi) = e^{-\check\alpha}(\chi). \tag{5.7}$$

Remark. Notice that in the functional equations for the case T non-split all that matters is whether $\tilde\chi$ is a ramified extension of χ and not which unramified/ramified extension it is (for instance, the cocharacter $e^{\frac{\check\alpha}{2}}$ does not appear). This is no coincidence: In this case the space $(\mathbf{T}\backslash \mathbf{SL}_2)(k)$ has two SL_2-orbits (isomorphic to each other), and therefore if instead of the distribution $S_{\tilde\chi^{-1}\nu^{-1}}$, given by an integral over all open B_2-orbits, we considered the restriction of that distribution to a single SL_2-orbit, then its Fourier transform should also be supported on the same SL_2-orbit. But these distributions which are supported on one SL_2-orbit do not "see" the extension of χ to the whole A_Y, but only to a subgroup of it, and one can check that that subgroup can detect ramification (it contains $\mathbf{A}_Y(\mathfrak{o})$) but not the precise extension of χ.

5.8. **Case N:.** Notice that for an one-dimensional torus $\mathbf{T}$ which is a spherical subgroup of $\mathbf{L}_\alpha$, if $\mathcal{N}_{\mathbf{L}_\alpha}(\mathbf{T}) \ne \mathcal{Z}_{\mathbf{L}_\alpha}(\mathbf{T})$ then $\mathbf{T}\subset \mathbf{L}'_\alpha$, and therefore in case N the variety $\mathbf{H}^0_\alpha\backslash\mathbf{L}_\alpha$ is of type T with associated cocharacters $\check v_{D_{1/2}} = \frac{\check\alpha}{2}$ if $\mathbf{H}^0_\alpha$ is split. To compute the functional equations for $\Delta_{\tilde\chi,\zeta}$ (§3.6), we first describe how to choose the point $\xi\in Y_\zeta$. Notice that Y_ζ may not contain integral points, therefore $\mathbf{H}^0_\alpha$ cannot always be assumed to be a smooth subgroup scheme over $\mathfrak{o}$ – hence, not all of the cases which we will encounter in case N have already been encountered in case T.

First of all, let $\xi_0\in\mathbf{Y}(\mathfrak{o})$ be a point in the "standard" B_0-orbit on $\mathbf{Y}(\mathfrak{o})$, and ζ_0 the corresponding coset of $A_{Y,\alpha}$. We identify $\mathbf{L}_\alpha/\mathcal{R}(\mathbf{L}_\alpha)$ with $\mathbf{PGL}_2$ and use ξ_0 to define a map: $\mathbf{YP}_\alpha\to\mathbf{X}_2 := \mathcal{N}(\mathbf{T})\backslash\mathbf{PGL}_2$ (for an integral torus $\mathbf{T}$ corresponding to the class of ξ_0). The latter is the space of binary quadratic forms modulo homotheties. Without loss of generality, ξ_0 maps to the quadratic form x^2+ay^2 for some $a\in -D(\zeta_0)\subset\mathfrak{o}^\times$. We choose $\xi\in Y_\zeta$ such that under the above map ξ maps to the homothety class of the quadratic form x^2+by^2 with $b\in -D(\zeta)$. (Recall from §3.5 that the discriminant $D(\zeta)$ is by its definition an element of $\mathfrak{o}\smallsetminus\mathfrak{p}^2$.)

From the definition of $\Delta_{\tilde\chi,\zeta} := \mathrm{ev}_1\circ\Delta^{Y,\alpha}_{\tilde\chi,\zeta}$ and Lemma 5.1.1 we deduce that the distribution $\Delta^2_{\tilde\chi,\zeta}$ on $U_\alpha\backslash L'_\alpha$ is given by:

$$\Delta^2_{\tilde\chi,\zeta} = \tilde\chi'(\xi)\eta(Q(x,y))dxdy \tag{5.8}$$

where, $\eta = \chi\delta^{-\frac12}\circ e^{\frac{\check\alpha}{2}}$ and Q is the quadratic form stabilized by $\mathbf{H}^0_2$ and $\tilde\chi'$ is the character of Lemma 5.1.1. (The factor $\tilde\chi'(\xi)$ is non-canonical, as is the distribution $\Delta^2_{\tilde\chi,\zeta}$, cf. §5.1; what is important is how this factor varies as $\tilde\chi$ varies.) By our choice of point ξ, up to an integral change of coordinates and up to a constant we have $Q(x,y)=x^2+\varpi y^2$, where $\varpi=-D(\zeta)$.

To determine the functional equations for $\Delta^2_{\tilde\chi,\zeta}$, if ζ corresponds to an integral torus, we can use our computations for the corresponding cases of type T, with $\check v_D=\frac{\check\alpha}{2}$. However, notice that now there is an extra factor of

$$\frac{\tilde\chi'(\xi)}{e^{-\alpha}\cdot^{w_\alpha}\tilde\chi'(\xi)} = \tilde\chi\circ e^{-\frac{\check\alpha}{2}}\left(\frac{D(\zeta)}{D(\zeta_0)}\right).$$

In the above, we took into account that if $\tilde\chi'$ is the character used in the expression (5.8) for $\Delta^2_{\tilde\chi,\zeta}$, then the character in the expression for $\Delta^2_{{}^{w_\alpha}\tilde\chi,\zeta}$ is $e^{-\alpha}\cdot^{w_\alpha}\tilde\chi'$. But

the quotient $\frac{\tilde\chi'(\xi)}{e^{-\alpha}\,({}^{w_\alpha}\tilde\chi')(\xi)}$ can be written as $\frac{\tilde\chi}{{}^{w_\alpha}\tilde\chi}$ (which in this case is a character of A_Y). By the definition of ξ, if $\bar\xi$ denotes its image in A_Y we get:

$$\frac{\tilde\chi}{{}^{w_\alpha}\tilde\chi}(\xi) = \tilde\chi\left(\frac{\bar\xi}{{}^{w_\alpha}\bar\xi}\right) = \tilde\chi\circ e^{-\frac{\check\alpha}{2}}\left(\frac{b}{a}\right) = \tilde\chi\circ e^{-\frac{\check\alpha}{2}}\left(\frac{D(\zeta)}{D(\zeta_0)}\right).$$

Hence, if ζ corresponds to a split integral torus and $\tilde\chi$ is $\frac{\check\alpha}{2}$-unramified then

$$\boxed{b^Y_{w_\alpha}(\tilde\chi,\zeta) = \left(\frac{1-q^{-\frac12}e^{-\frac{\check\alpha}{2}}(\tilde\chi)}{1-q^{-\frac12}e^{\frac{\check\alpha}{2}}(\tilde\chi)}\right)^2}, \tag{5.9}$$

if ζ corresponds to a split integral torus and $\tilde\chi$ is $\frac{\check\alpha}{2}$-ramified then

$$\boxed{b^Y_{w_\alpha}(\tilde\chi,\zeta) = \tilde\chi\circ e^{\frac{-\check\alpha}{2}}\left(\frac{D(\zeta)}{D(\zeta_0)}\right)e^{-\check\alpha}(\chi)}, \tag{5.10}$$

if ζ corresponds to a non-split integral torus and $\tilde\chi$ is $\frac{\check\alpha}{2}$-unramified then

$$\boxed{b^Y_{w_\alpha}(\tilde\chi,\zeta) = \frac{1-q^{-1}e^{-\check\alpha}(\chi)}{1-q^{-1}e^{\check\alpha}(\chi)}}, \tag{5.11}$$

while if ζ corresponds to a non-split integral torus and $\tilde\chi$ is $\frac{\check\alpha}{2}$-ramified then

$$\boxed{b^Y_{w_\alpha}(\tilde\chi,\zeta) = \tilde\chi\circ e^{\frac{-\check\alpha}{2}}\left(\frac{D(\zeta)}{D(\zeta_0)}\right)e^{-\check\alpha}(\chi)}. \tag{5.12}$$

There remains to compute the equations for the case that ζ does not correspond to an integral torus. Here we will denote b by ϖ to remind of the fact that $\varpi = -D(\zeta)$ will be a uniformizing element in k, uniquely defined up to multiplication by $(o^\times)^2$. Hence $Q(x,y) = x^2+\varpi y^2$.

Now we notice the following: The form $Q(x,y)$ takes values not in the whole of k but only in $\{1,\varpi\}\cdot(\mathfrak{o}^\times)^2$. This implies that the values of $\eta(Q(x,y))$ will be the same for two characters η agreeing on $\{1,\varpi\}\cdot(\mathfrak{o}^\times)^2$, and since η^2 is unramified there exists an unramified such character. To compute the Fourier transforms, we may therefore without loss of generality assume that η is unramified. The reader should compare this with Remark 5.7: Here it will not matter whether $\tilde\chi$ is ramified or not, but only the value of its pull-back via $e^{\frac{\check\alpha}{2}}$ at $\varpi = -D(\zeta)$.

Hence, we may write $\eta(\bullet) = |\bullet|^{\frac{s}{2}}$ where $q^{\frac{s}{2}} = \tilde\chi\delta^{-\frac12}\circ e^{-\frac{\check\alpha}{2}}(-D(\zeta)) = q^{-\frac12}\tilde\chi\circ e^{-\frac{\check\alpha}{2}}(-D(\zeta))$. Now we compute:

$$\left\langle \Delta^2_{\tilde\chi,\zeta}, 1_{\mathfrak{o}^2}\right\rangle = \tilde\chi'(\xi)\int_{\mathfrak{o}^2}|x^2+\varpi y^2|^{\frac{s}{2}}dxdy =$$

$$= \tilde\chi\nu\delta^{-\frac12}(\bar\xi)\left(\int_{|x|<|y|\le 1} q^{-\frac{s}{2}}|y|^s dxdy + \int_{|y|<|x|\le 1}|x|^s dxdy + \int_{|x|=|y|\le 1}|x|^s dxdy\right) =$$

$$= \tilde\chi\nu\delta^{-\frac12}(\bar\xi)\left((1+q^{-\frac{s}{2}})\frac{q^{-1}(1-q^{-1})}{1-q^{-s-2}} + \frac{(1-q^{-1})^2}{1-q^{-s-2}}\right) =$$

$$= \tilde\chi\nu\delta^{-\frac12}(\bar\xi)\frac{(1-q^{-1})(1+q^{\frac{-s-2}{2}})}{1-q^{-s-2}} = \tilde\chi\nu\delta^{-\frac12}(\bar\xi)\frac{1-q^{-1}}{1-q^{\frac{-s-2}{2}}}.$$

Similarly, the integral of $\Delta^2_{w_\alpha\chi,\zeta} = e^{-\alpha}\cdot {}^{w_\alpha}\tilde{\chi}'(\xi)|x^2+\varpi y^2|^{-\frac{s}{2}-1}dxdy$:

$$\left\langle \Delta^2_{w_\alpha\tilde{\chi},\zeta}, 1_{\mathfrak{o}^2}\right\rangle = e^{-\alpha}\cdot {}^{w_\alpha}\tilde{\chi}'(\xi)\frac{1-q^{-1}}{1-q^{\frac{s}{2}}}.$$

The quotient of the two is:

$$b^Y_{w_\alpha}(\tilde{\chi},\zeta) = \tilde{\chi}\circ e^{\frac{-\check{\alpha}}{2}}\left(\frac{D(\zeta)}{D(\zeta_0)}\right)\frac{1-q^{\frac{s}{2}}}{1-q^{\frac{-s-2}{2}}}.$$

Therefore:

$$\boxed{b^Y_{w_\alpha}(\tilde{\chi},\zeta) = \tilde{\chi}\circ e^{\frac{-\check{\alpha}}{2}}\left(\frac{D(\zeta)}{D(\zeta_0)}\right)\frac{1-q^{-\frac{1}{2}}\tilde{\chi}\circ e^{-\frac{\check{\alpha}}{2}}(-D(\zeta))}{1-q^{-\frac{1}{2}}\tilde{\chi}\circ e^{\frac{\check{\alpha}}{2}}(-D(\zeta))}} \tag{5.13}$$

Part 2. The formula with L-values

6. Simple spherical reflections

From now on we assume that $\mathbf{X}$ is a spherical variety such that there are no reflections of type N, in other words there does not exist an orbit Y of maximal rank and a simple root α such that (Y,α) is of type N. Moreover, the character $\delta_{(X)}^{-\frac{1}{2}}\tilde{\chi}$ of A_X is assumed to be unramified. In the case of a parabolically induced $\mathbf{X}$ with a non-trivial character $\Psi=\psi\circ\Lambda$, we assume that Λ (or Ψ) is *generic*: that means that Λ belongs to the open orbit of the corresponding Levi on the additive characters of the unipotent radical. (These are the most interesting cases.)

6.1. The root system of a spherical variety. Up to now we have developed an algorithm for calculating the constants in formula (4.2) of Theorem 4.2.2 by calculating the functional equations for all orbits Y of maximal rank and simple roots α *of* G. However, the formula really depends only on the functional equations for $\Delta_{\tilde{\chi}}$, the morphism defined by the open $\mathbf{B}$-orbit. Those are parametrized by the little Weyl group W_X.

It is known that the faithful action of W_X on $\mathcal{X}(\mathbf{X})\otimes\mathbb{Q}$ (equivalently on $\mathcal{Q}=\operatorname{Hom}(\mathcal{X}(\mathbf{X}),\mathbb{Q})$) is generated by reflections, and the cone $\mathcal{V}\subset\mathcal{Q}$ of invariant valuations is a fundamental domain for its action on $\mathcal{Q}$. In fact W_X is the Weyl group of a root system, and the rank of this root system is called the *rank* of the spherical variety $\mathbf{X}$. (We will use this term only when it is clear that we are not refering to the other notion of rank – namely, the rank of $\mathbf{X}$ as a $\mathbf{B}$-variety.) This root system is defined as follows [Kn96]: One considers in $\mathcal{X}(\mathbf{A}_X)\otimes\mathbb{Q}$ the cone (negative) dual to $\mathcal{V}$, i.e. the set $\{\chi\in\mathcal{X}(\mathbf{X})\otimes\mathbb{Q}|\left\langle v,\chi\right\rangle\le 0 \text{ for every } v\in\mathcal{V}\}$. This is a strictly convex cone, and the simple roots of the root system are the generators for the intersections of $\mathcal{X}(\mathbf{X})$ with its extremal rays. It seems that this root system is not quite the correct one for the purposes of representation theory, therefore we describe in [SV] (and recall below) a variant Φ_X of that, on the same vector space and with the same Weyl group, but with roots of different length. The dual root system $\check{\Phi}_X$ is expected to be the root system of the dual group $\check{G}_{GN}$ that Gaitsgory and Nadler [GN] have associated to the spherical variety. In [SV] we show that if there are no reflections of type N then the data $(\mathcal{X}(\mathbf{X})^*,\check{\Phi}_X,\mathcal{X}(\mathbf{X}),\Phi_X)$ form a root datum and hence define uniquely up to isomorphism a complex group $\check{G}_X$ with a canonical maximal torus A_X^* and root system $\check{\Phi}_X$; it will be called "the dual group of $\mathbf{X}$".

For the case of $C^\infty(X, \mathcal{L}_\Psi)$, we explained in [Sa08] how to associate similar invariants based on Knop's definitions for non-spherical varieties. Notice the following:

6.1.1. **Lemma.** *Let $\Lambda : \mathbf{U}_{P^-} \to \mathbb{G}_a$ be an additive character of the unipotent radical of a parabolic $\mathbf{P}^-$, normalized by a spherical subgroup $\mathbf{M}$ of the Levi $\mathbf{L}$. We assume that $\mathbf{P}^-$ is opposite to a standard parabolic and $\mathbf{M}\cdot(\mathbf{L}\cap\mathbf{B})$ is open in $\mathbf{L}$. Then Λ is completely determined by its restriction to the simple root subgroups $\mathbf{U}_{-\alpha}$, $\alpha \in \Delta \smallsetminus \Delta_L$. A generic character is non-trivial on all those subgroups.*

Proof. For the first statement, it suffices to show that if Λ is zero on all those simple root subgroups, then it is trivial. Given that $[\mathbf{U}_{-\alpha}, \mathbf{U}_\beta] = 1$ for all distinct $\alpha, \beta \in \Delta$, the subgroups $\mathbf{U}_{-\alpha}$, $\alpha \in \Delta \smallsetminus \Delta_L$, are normalized by $\mathbf{B}_L := \mathbf{L} \cap \mathbf{B}$. In combination with the fact that $\mathbf{M}$ stabilizes Λ, $bm \cdot \Lambda$ is zero on those subgroups, for every $b \in \mathbf{B}_L$, $m \in \mathbf{M}$. But $\mathbf{B}_L\mathbf{M}$ is open in $\mathbf{L}$, hence, $l \cdot \Lambda$ is zero on all those subgroups. The only character with this property is the trivial one.

For the second claim, the same argument shows that if Λ is zero on one of the $\mathbf{U}_{-\alpha}$'s, then the same is true for $l \cdot \Lambda$, for every $l \in \mathbf{L}$. Such a Λ cannot be generic. □

Here is a description of the simple roots of Φ_X: First of all, among the elements of W_X the reflections $w_\gamma \in W_X$ corresponding to simple roots are completely determined by the requirement that the corresponding negative Weyl chamber be $\mathcal{V}$, which contains the image of the negative Weyl chamber of $\mathbf{G}$. Recall Brion's description of generators for $W_{(X)}$ (Theorem 2.3.2). We argue in [SV] that this description applies in particular to simple reflections $w_\gamma \in W_X$. Hence, to repeat, adding the case of a non-trivial $\mathcal{L}_\Psi$, every $w = w_\gamma$ can be written as $w = w_1^{-1} w_2 w_1$ where:

- ${}^{w_1}\mathring{X} =: Y$ with $\mathrm{codim}(Y) = l(w_1)$. (In the notation of 2.3, $w_1^{-1} \in W(\mathbf{Y})$.)
- w_2 is either of the following three:
  1. equal to w_α, where α is a simple root such that (Y, α) is of type T (or N, if we allow type N reflections), or
  2. equal to $w_\alpha w_\beta$, where α, β are two orthogonal simple roots which both lower Y to the same orbit Y', or
  3. equal to w_α, where α is a simple root such that (Y, α) is of type (U, ψ). (By our assumption that Ψ be generic and Lemma 6.1.1, in this case w_1 is trivial.)

For any path $\mathfrak{g}$ in Knop's diagram that corresponds to such a description of w_γ we will say that $\mathfrak{g}$ *leads* or *corresponds* to w_γ. Depending on which of the enumerated cases above appears:

1. The weight $\gamma = w_1^{-1}\alpha$ will be called a *spherical root of type T (resp. N)*.
2. The weight $\gamma = w_1^{-1}(\alpha + \beta)$ will be called a *spherical root of type G*.[9]
3. The weight $\gamma = \alpha = w_1^{-1}\alpha$ will be called a *spherical root of type* (U, ψ).

We show in [SV] that the resulting root system Φ_X, which is a modification of that defined by Knop, is well-defined and does not depend on choices. The set of simple spherical roots will be denoted by Δ_X (not to be confused with $\Delta(\mathbf{X})$), and the set of positive spherical roots by Φ_X^+. The rank of this root system is called the

[9]The name "type G" originates from the fact that this is the spherical root for the variety $\mathbf{SL}_2 \backslash \mathbf{SL}_2 \times \mathbf{SL}_2$. This shouldn't cause any confusion with "reflections of type G", since the latter do not generate any spherical roots.

rank of the spherical variety. Notice that in case T (and N) the spherical co-root $\check{\gamma}$ is well defined not only as an element of the dual root system $\check{\Phi}_X$ of Φ_X, but also as an element of $\check{\Phi}$. We will see that the spherical roots (or rather, their co-roots) naturally appear in our formula for Hecke eigenvectors. It is clearly enough to compute the proportionality factors b_{w_γ} for all simple roots γ in Δ_X. For those, we introduce the following notation: $\mathbf{P}_\gamma, \mathbf{L}_\gamma$ are the standard parabolic and Levi $\mathbf{P}_{\mathrm{supp}(\gamma)}, \mathbf{L}_{\mathrm{supp}(\gamma)}$, and $\mathbf{X}_\gamma$ the spherical variety $\mathring{\mathbf{X}}\mathbf{P}_\gamma/\mathbf{U}_{P_\gamma}$ for $\mathbf{L}_\gamma$.

Finally, for the discussion which follows recall that $\mathbf{B}$-invariant absolutely irreducible divisors in $\mathbf{X}_{\bar{k}}$ are called "colors". We notice that over a non-algebraically closed field k a color may not be defined over k, as happens in the case of $\mathbf{T}\backslash\mathbf{SL}_2$ with $\mathbf{T}$ a non-split torus. In fact, this is essentially the only case where a color is not defined over k. More precisely, every color is contained in $(\mathring{\mathbf{X}}\mathbf{P}_\alpha)_{\bar{k}}$ for some simple root α, and if $(\mathring{\mathbf{X}}, \alpha)$ is of type U or T split then the absolutely irreducible divisors in $(\mathring{\mathbf{X}}\mathbf{P}_\alpha)_{\bar{k}}$ are defined over k.

6.2. **The values of the cocycles for simple spherical roots.** For the statement of the results it is convenient to introduce the constants B_w (for $w \in W_X$), defined as:

$$B_w(\tilde{\chi}) = \prod_{\check{\alpha}\in\check{\Phi}^+, w\check{\alpha}<0} (-e^{\check{\alpha}}(\chi)) b_w(\delta_{(X)}^{\frac{1}{2}}\tilde{\chi}) \tag{6.1}$$

The B_w are cocycles: $W_X \to \mathbb{C}(A_X^*)$. Using those, Theorem 4.2.2 becomes:

$$\Omega_{\delta_{(X)}^{\frac{1}{2}}\tilde{\chi}}(x) = \delta_{P(X)}^{-\frac{1}{2}}(x) \sum_{w\in W_X} B_w(\tilde{\chi})\, {}^w\tilde{\chi}(x). \tag{6.2}$$

Our goal at this point is to compute B_w for $w = w_\gamma$, where w_γ is a simple spherical root. We are aiming at the following:

6.2.1. **Statement.** *Let $\mathfrak{g}$ be a path in Knop's diagram corresponding to the simple reflection w_γ. Let $\check{\theta}$ be the valuation induced by the codimension-one orbit in the path, or if $\mathring{\mathbf{X}}$ is the only node in the path, i.e. γ is in Δ and hence a spherical root of type T, let $\check{\theta}$ be the valuation induced by a color in $\mathring{\mathbf{X}}\mathbf{P}_\gamma$. In any case, $\check{\theta}$ is viewed as an element of* $\mathrm{Hom}(\mathcal{X}(\mathbf{X}), \mathbb{Z})$. *Then, according to the type of the root γ we have:*

- *If γ is of type G, then $\check{\theta} = \check{\gamma}$ and:*

$$B_{w_\gamma}(\tilde{\chi}) = -e^{\check{\gamma}} \frac{1 - q^{-\langle\check{\gamma},\rho_{P(X)}\rangle} e^{-\check{\gamma}}}{1 - q^{-\langle\check{\gamma},\rho_{P(X)}\rangle} e^{\check{\gamma}}}(\chi). \tag{6.3}$$

- *If γ is of type T split, then $\langle\check{\theta}, \gamma\rangle = 1$ and:*

$$B_{w_\gamma}(\tilde{\chi}) = -e^{\check{\gamma}} \frac{1 - q^{-\langle\check{\theta},\rho_{P(X)}\rangle} e^{-\check{\theta}}}{1 - q^{\langle\check{\theta},\rho_{P(X)}\rangle - \langle\check{\rho},\gamma\rangle} e^{\check{\theta}}} \frac{1 - q^{\langle\check{\theta},\rho_{P(X)}\rangle - \langle\check{\rho},\gamma\rangle} e^{w_\gamma\check{\theta}}}{1 - q^{-\langle\check{\theta},\rho_{P(X)}\rangle} e^{-w_\gamma\check{\theta}}}(\tilde{\chi}). \tag{6.4}$$

- *If γ is of type T non-split, then $\check{\theta} = \frac{\check{\gamma}}{2}$ and:*

$$B_{w_\gamma}(\tilde{\chi}) = -e^{\check{\gamma}} \frac{1 - q^{-\langle\check{\theta},\rho_{P(X)}\rangle} e^{-\check{\theta}}}{1 + q^{\langle\check{\theta},\rho_{P(X)}\rangle - \langle\check{\rho},\gamma\rangle} e^{\check{\theta}}} \frac{1 + q^{\langle\check{\theta},\rho_{P(X)}\rangle - \langle\check{\rho},\gamma\rangle} e^{w_\gamma\check{\theta}}}{1 - q^{-\langle\check{\theta},\rho_{P(X)}\rangle} e^{-w_\gamma\check{\theta}}}(\tilde{\chi}). \tag{6.5}$$

- *If γ is of type (U,ψ), then*

$$B_{w_\gamma}(\tilde{\chi}) = -e^{\tilde{\gamma}}(\chi). \tag{6.6}$$

In case T, the exponents $-\left\langle \check{\theta}, \rho_{P(X)} \right\rangle$ and $\left\langle \check{\theta}, \rho_{P(X)} \right\rangle - \langle \check{\rho}, \gamma \rangle$ are equal, unless possibly if $\check{\theta} = \frac{\check{\gamma}}{2}$. In any case, these exponents are negative half-integers.

Remarks. (1) Recall that $\delta_{P(X)} \in A_X^*$ (3.5) and therefore the exponential of $\left\langle \tilde{\gamma}, \rho_{P(X)} \right\rangle$ makes sense.
(2) A priori it is not clear that being of type T split or non-split is a well-defined property of γ, independent of the path. A posteriori, this is true if the above statement is true.

We can only prove this statement by reducing to certain low-rank spherical varieties, and we have performed the corresponding computation only for classical groups:

6.2.2. **Theorem.** *Let $\mathbf{X}$ be a spherical variety without type-N roots and such that for every $\gamma \in \Delta_X$ the Levi $\mathbf{L}_\gamma$ is a classical group. Then $\mathbf{X}$ satisfies Statement 6.2.1.*

6.3. **Glueing paths from simple varieties.** Our goal for the rest of this section is to prove Theorem 6.2.2. Let $\gamma \in \Delta_X$, and choose a path $\mathfrak{g}$ in Knop's graph $\mathfrak{G}$ which corresponds to w_γ. Let $\mathbf{Z} \neq \mathring{\mathbf{X}}$ be a node in the path $\mathfrak{g}$ and let e_1, e_2 be edges of the path having $\mathbf{Z}$ as one of their endpoints, and such that e_1 raises $\mathbf{Z}$. (Hence, e_2 either lowers $\mathbf{Z}$ or goes from $\mathbf{Z}$ to itself, in which case $\mathbf{Z}$ is the orbit of smallest dimension in the path.) Let δ, ϵ be the simple roots of $\mathbf{G}$ labelling e_1 and e_2, and consider the spherical variety $\mathbf{ZP}_{\delta\epsilon}/\mathcal{R}(\mathbf{P}_{\delta\epsilon})$ for the group $\mathbf{L}_{\delta\epsilon}$. If $\mathbf{G}_1, \mathbf{G}_2, \dots$ are reductive groups of rank two, and $\mathbf{X}_1, \mathbf{X}_2, \dots$ spherical varieties thereof, we will say that the path $\mathfrak{g}$ is "glued" from $\mathbf{X}_1, \mathbf{X}_2, \dots$ if all spherical varieties $\mathbf{ZP}_{\delta\epsilon}/\mathcal{R}(\mathbf{P}_{\delta\epsilon})$ obtained this way are contained in the list of $\mathbf{X}_1, \mathbf{X}_2, \dots$. As a matter of language, if the path begins and ends at $\mathring{\mathbf{X}}$ then the path $\mathfrak{g}$ is glued from any list of such spherical varieties.

We have:

6.3.1. **Proposition.** *Assume that there is a path $\mathfrak{g}$ corresponding to w_γ which over the algebraic closure is glued out of the following varieties:*

$$\mathbf{PGL}_2 \setminus \mathbf{PGL}_2 \times \mathbf{PGL}_2, \mathbf{GL}_2 \setminus \mathbf{SL}_3,$$
$$\mathbf{Sp}_2 \times \mathbf{Sp}_2 \setminus \mathbf{Sp}_4, (\mathbb{G}_m \ltimes \mathbb{G}_a) \times \mathbf{Sp}_2 \setminus \mathbf{Sp}_4, \mathbb{G}_m \times \mathbf{Sp}_2 \setminus \mathbf{Sp}_4,$$
$$\mathbf{P}_\alpha \setminus \mathbf{SL}_3 \text{ (where } \alpha \text{ is a simple root) and}$$
$$\mathbf{P}_\alpha \setminus \mathbf{Sp}_4 \text{ (where } \alpha \text{ is the long simple root).}$$

Then $\mathfrak{g}$ satisfies Statement 6.2.1, except possibly for the last assertion (on the exponents).

Remark. The first two rows in the list consist of all spherical varieties without spherical roots of type N, for simply connected groups of rank two other than $\mathbf{G}_2$, and with the property that both simple roots are in the support of a spherical root (see Theorem 6.10.1 below). Therefore, unless $\mathring{\mathbf{X}}$ is the only node of the path $\mathfrak{g}$, the bottom part of the path will necessarily consist of one of these varieties. Hence, if the group has no $\mathbf{G}_2$-factors, the content of the assumption is contained in the next lines, where one restricts the horospherical varieties which can glue the path.

Moreover:

6.3.2. **Proposition.** *If $\gamma \in \Delta_X$ with $\mathbf{L}_{\mathrm{supp}(\gamma)}$ a classical group then for any path $\mathfrak{g}$ corresponding to w_γ there is a path $\mathfrak{g}'$ corresponding to w_γ such that $\mathfrak{g}$ and $\mathfrak{g}'$ have the same first edge and $\mathfrak{g}'$ satisfies the assumptions of Proposition 6.3.1.*

Moreover, in case T the exponents $-\left\langle \check\theta, \rho_{P(X)}\right\rangle$ and $\left\langle\check\theta, \rho_{P(X)}\right\rangle - \langle\check\rho, \gamma\rangle$ are equal, unless possibly if $\check\theta = \frac{\check\gamma}{2}$. In any case, these exponents are negative half-integers.

Since Statement 6.2.1 only depends on the first edge of $\mathfrak{g}$, this proves Theorem 6.2.2. The goal of the rest of this section is to prove Propositions 6.3.1 and 6.3.2.

6.4. **Computation for the simple varieties.** Proposition 6.3.1 will follow from the following:

6.4.1. **Proposition.** *Assume that the path $\mathfrak{g}$ in Knop's graph, corresponding to w_γ for a simple spherical root γ, satisfies the assumptions of Proposition 6.3.1. Let $\mathbf{Z}$ be a node of $\mathfrak{g}$ other than the lowest one (in particular, we assume that $\mathring{\mathbf{X}}$ is not the only node of $\mathfrak{g}$), w the element of the Weyl group corresponding to the path below $\mathbf{Z}$. Hence $w = w_1^{-1}w_2w_1$, where w_1 lowers $\mathbf{Z}$ to an orbit $\mathbf{Y}$ and $w_2 = w_\beta w_{\tilde\beta}$ or $w_2 = w_\beta$, according as γ is of type G or T. Let α be the label of an edge in $\mathfrak{g}$ which lowers $\mathbf{Z}$. Let ${}^{‘}b_w^Z(\tilde\chi) = \prod_{\check\epsilon\in\check\Phi^+, w\check\epsilon<0}(-e^{\check\epsilon}(\chi))b_w^Z(\tilde\chi)$. Then:*

- *If the root γ is of type G then*

$$
{}^{‘}b_w^Z(\tilde\chi) = -e^{\check\gamma'}(\chi)\frac{1-q^{-1}e^{-\check\alpha}}{1-q^{-1}e^{-w\check\alpha}}(\chi) \tag{6.7}
$$

  where $\check\gamma' = w_1^{-1}\check\beta$. Moreover, we have that $\check\alpha|_{\mathcal{X}(\mathbf{Z})} = \check\gamma'|_{\mathcal{X}(\mathbf{Z})}$. (Recall that as an element of $\mathcal{X}(\mathbf{Z})^*$, this does not depend on the choice between β and $\tilde\beta$, and similarly $e^{\check\gamma'}(\chi)$ does not depend on this choice for characters χ of A for which $\Delta_{\tilde\chi}^Z$ is defined.)
- *If the root γ is of type T split, then*

$$
{}^{‘}b_w^Z(\tilde\chi) = -e^{\check\gamma'}(\chi)\frac{1-q^{-1}e^{-\check\alpha}}{1-q^{1-\langle\check\rho,\gamma'\rangle}e^{\check\alpha}}\frac{1-q^{1-\langle\check\rho,\gamma'\rangle}e^{w\check\alpha}}{1-q^{-1}e^{-w\check\alpha}}(\chi) \tag{6.8}
$$

  where $\gamma' = w_1^{-1}\beta$, $\check\gamma' = w_1^{-1}\check\beta$. Moreover, in this case $\gamma' \in \mathcal{X}(\mathbf{Z})$ and $\langle\check\alpha, \gamma'\rangle = 1$.
- *If the root γ is of type T non-split, then*

$$
{}^{‘}b_w^Z(\tilde\chi) = -e^{\check\gamma'}(\chi)\frac{1-q^{-1}e^{-\check\alpha}}{1+q^{1-\langle\check\rho,\gamma'\rangle}e^{\check\alpha}}\frac{1+q^{1-\langle\check\rho,\gamma'\rangle}e^{w\check\alpha}}{1-q^{-1}e^{-w\check\alpha}}(\chi) \tag{6.9}
$$

  where $\gamma' = w_1^{-1}\beta$, $\check\gamma' = w_1^{-1}\check\beta$. Moreover, in this case $\gamma' \in \mathcal{X}(\mathbf{Z})$ and $\check\alpha|_{\mathcal{X}(\mathbf{X})} = \frac{\check\gamma'}{2}$.

Let us first see why this proves Proposition 6.3.1:

Proof of Proposition 6.3.1. First of all, if $(\mathring{\mathbf{X}}, \alpha)$ is of type T split, then setting $\delta_{(U_{P_\alpha})_\xi} = \delta_{(X)}$ and substituting $\tilde\chi$ by $\delta_{(X)}^{\frac12}\tilde\chi$ in formula (5.4) we get:

$$
e^{\check\upsilon_D}(\delta_{(X)}^{\frac12}\tilde\chi\delta^{\frac12}\delta_{(X)}^{-1}) = e^{\check\upsilon_D}(\tilde\chi\delta_{P(X)}^{\frac12}),
$$

and similarly for $\check{v}_{D'}$. Given the property that $B_{w_\alpha}(\delta^{\frac{1}{2}}_{P(X)}) = 0$ (by Lemma 4.2.3)[10], it follows that $e^{\check{v}_D}(\delta_{P(X)}) = q^{-1}$ for a suitable naming of $\mathbf{D}, \mathbf{D}'$, and the fact that $\check{v}_{D'} = \check{\alpha} - \check{v}_D$ and $e^{\check{\alpha}}(\delta_{P(X)}) = e^{\check{\alpha}}(\delta) = q^{-2}$ implies that the same is true for $\check{v}_{D'}$ in place of $\check{v}_D$. This verifies (6.4) in this case.

Similarly, if $(\mathring{\mathbf{X}}, \alpha)$ is of type T non-split, then (6.5) follows from (5.6) and if $(\mathring{\mathbf{X}}, \alpha)$ is of type (U, ψ) then (6.6) follows from (5.3).

In all other cases, if α is the label of the first edge in the path $\mathfrak{g}$ then $(\mathring{\mathbf{X}}, \alpha)$ is of type U and the last proposition applies. Notice that $\mathcal{X}(\mathbf{X}) \perp \Delta(\mathbf{X})$, therefore $e^{\check{\gamma}}(\chi\delta^{\frac{1}{2}}_{(X)}) = e^{\check{\gamma}}(\chi)$. From the fact that $\delta_{P(X)} \in A^*_X$ we get in case of a root of type G that: $e^{\check{\alpha}}(\delta^{\frac{1}{2}}_{(X)}) = e^{\check{\alpha}}(\delta^{\frac{1}{2}}\delta^{-\frac{1}{2}}_{P(X)}) = q^{-1}e^{\check{\gamma}}(\delta^{-\frac{1}{2}}_{P(X)})$ and this proves (6.3). Finally, (6.4) and (6.5) follow from (6.8), resp. (6.9), because $\check{\theta} = \check{\alpha}$ and $e^{\check{\alpha}}(\delta^{\frac{1}{2}}_{(X)}\delta^{\frac{1}{2}}_{P(X)}) = q^{-1}$. □

Proof of Proposition 6.4.1. We prove it by induction on the dimension of $\mathbf{Z}$. We start with the case G. First of all, if $\mathbf{Z}$ is equal to $\mathbf{Y}$ and $\alpha = \beta, \tilde{\beta}$ are the two orthogonal roots lowering $\mathbf{Z}$ to another orbit then we have computed in Example 5.4.1 that in this case $\check{\gamma}' = \check{\alpha}$ and:

$$`b^Z_w(\tilde{\chi}) = -e^{\check{\alpha}}\frac{1 - q^{-1}e^{-\check{\alpha}}}{1 - q^{-1}e^{\check{\alpha}}}(\chi).$$

Now assume that $\mathbf{Z}$ is lowered by α to some vertex $\mathbf{V}$ of $\mathfrak{g}$, and let β be an edge lowering $\mathbf{V}$. By the induction hypothesis,

$$`b^V_{w_\alpha w w_\alpha}({}^{w_\alpha}\tilde{\chi}) = -e^{w_\alpha\check{\gamma}'}\frac{1 - q^{-1}e^{-\check{\beta}}}{1 - q^{-1}e^{\check{\beta}}}({}^{w_\alpha}\chi).$$

Using our functional equations for roots of type U, we compute:

$$`b^Z_w(\tilde{\chi}) = -e^{\check{\gamma}'}\frac{1 - q^{-1}e^{-\check{\alpha}}}{1 - e^{-\check{\alpha}}}(\chi) \cdot \frac{1 - q^{-1}e^{-w_\alpha\check{\beta}}}{1 - q^{-1}e^{w_\alpha\check{\beta}}}(\chi) \cdot \frac{1 - e^{-w\check{\alpha}}}{1 - q^{-1}e^{-w\check{\alpha}}}(\chi).$$

To complete the proof in this case, it suffices to prove that if $\mathbf{V}\mathbf{P}_{\alpha\beta}/\mathcal{R}(\mathbf{P}_\alpha\beta)$ is among the list of varieties in the assumptions of the proposition, then:

$$e^{-\check{\alpha}}(\chi) = q^{-1}e^{-w_\alpha\check{\beta}}(\chi) \tag{6.10}$$

This will be proved via a *case-by-case* computation below. Finally, since by the induction assumption we have $\check{\beta}|_{\mathcal{X}(\mathbf{V})} = w_\alpha\check{\gamma}'|_{\mathcal{X}(\mathbf{V})}$ and $\mathcal{X}(\mathbf{Z}) = w_\alpha\mathcal{X}(\mathbf{V})$ it follows from (6.10) that $\check{\alpha}|_{\mathcal{X}(\mathbf{Z})} = \check{\gamma}'|_{\mathcal{X}(\mathbf{Z})}$.

In cases T split and non-split, we will see in the *case-by-case* analysis that the statement of Proposition 6.4.1 is correct when $\mathbf{Z}$ =the node $\mathbf{V}$ of $\mathfrak{g}$ which is immediately higher than the lowest node. Now, if the hypothesis is satisfied by a node $\mathbf{V}$ with β a root lowering $\mathbf{V}$ and α a root raising it to a node $\mathbf{Z}$, then (in the case T split, the non-split case being completely analogous):

$$`b^Z_w(\tilde{\chi}) = \frac{1 - q^{-1}e^{-\check{\alpha}}}{1 - e^{-\check{\alpha}}}(\chi) \cdot (-e^{w_\alpha\check{\gamma}'})\frac{1 - q^{-1}e^{-\check{\beta}}}{1 - q^{1-\langle\check{\rho}, w_\alpha\gamma'\rangle}e^{\check{\beta}}} \cdot$$
$$\cdot\frac{1 - q^{1-\left\langle\check{\rho}, w_\alpha\gamma'\right\rangle}e^{(w_\alpha w w_\alpha)\check{\beta}}}{1 - q^{-1}e^{-(w_\alpha w w_\alpha)\check{\beta}}}({}^{w_\alpha}\chi) \cdot \frac{1 - e^{-w\check{\alpha}}}{1 - q^{-1}e^{-w\check{\alpha}}}(\chi).$$

[10]In case we are considering a line bundle $\mathcal{L}_\Psi$ over X, we still have $B_{w_\alpha}(\delta^{\frac{1}{2}}_{P(X)}) = 0$ by using, for instance, the variety $\mathbf{X}$ without the character Ψ.

Once again, it suffices to prove (6.10). Indeed, then the factor $1-e^{-\check\alpha}(\chi)$ cancels $1-q^{-1}e^{-\check\beta}({}^{w_\alpha}\chi)$. Moreover, since $w_\alpha\gamma' \in \mathcal{X}(\mathbf{V}) \iff \gamma' \in \mathcal{X}(\mathbf{Z})$ we get $\langle\gamma',\check\alpha\rangle = \left\langle\gamma', w_\alpha\check\beta\right\rangle = \left\langle w_\alpha\gamma',\check\beta\right\rangle = 1$ by (6.10) and the induction hypothesis. By (6.10) we also have $e^{-\check\alpha}({}^{w^{-1}}\chi) = q^{-1}e^{-w_\alpha\check\beta}({}^{w^{-1}}\chi)$ and therefore the factor $1-e^{-w\check\alpha}(\chi)$ cancels the factor $1-q^{-1}e^{-(w_\alpha w w_\alpha)\check\beta}({}^{w_\alpha}\chi)$. Finally, we have

$$q^{1-\left\langle w_\alpha\gamma',\check\rho\right\rangle}e^{\check\beta}({}^{w_\alpha}\chi) = q^{1-\left\langle\gamma',\check\rho\right\rangle}q^{\left\langle\gamma',\check\alpha\right\rangle}e^{w_\alpha\check\beta}(\chi) = q^{1-\left\langle\gamma',\check\rho\right\rangle}e^{\check\alpha}(\chi)$$

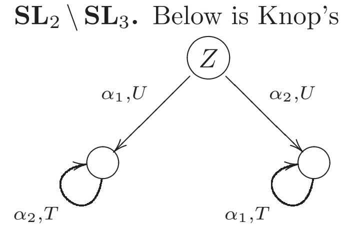
since $\gamma'\in\mathcal{X}(\mathbf{Z})$ and hence $\langle\check\alpha,\gamma'\rangle = \left\langle\frac{\check\gamma}{2},\gamma\right\rangle = 1$. This completes, up to the case-by-case analysis that follows, the proof of the proposition. ☐

Now we study one-by-one the "simple" varieties of Proposition 6.3.1, in order to establish the remaining points of its proof. More precisely, let $\mathbf{Z}$ be an orbit in a given path $\mathfrak{g}$ and α,β two simple roots such that $\mathbf{ZP}_{\alpha\beta}/\mathcal{R}(\mathbf{U}_{P_{\alpha\beta}})$ is one of the varieties listed in Proposition 6.3.1. Let $\mathbf{H}_{\alpha\beta}$ be the stabilizer mod $\mathbf{U}_{P_{\alpha\beta}}$ of a point on $\mathbf{ZP}_{\alpha\beta}$ (hence, by definition, $\mathbf{H}_{\alpha\beta}$ is a subgroup of the reductive quotient $\mathbf{L}_{\alpha\beta}$), and set $\mathbf{H}'=[\mathbf{H}_{\alpha\beta},\mathbf{H}_{\alpha\beta}]$ and $\mathbf{G}'=[\mathbf{L}_{\alpha\beta},\mathbf{L}_{\alpha\beta}]$. We can prove the remaining statements of Proposotion 6.4.1 by looking at the variety $\mathbf{H}'\backslash\mathbf{G}'$; the latter may not be spherical for $\mathbf{G}'$, in which case we will treat it as a spherical variety for $\mathbf{G}'\times$ the connected component of $\mathcal{N}_{\mathbf{L}_{\alpha\beta}}(\mathbf{H}_{\alpha\beta})\cap\mathbf{G}'$.

Recall that the formulas of section 5 (and in particular (5.4)) do not depend just on the space $\mathbf{H}'\backslash\mathbf{G}'$ but also on some modular characters of subgroups of $\mathbf{P}_{\alpha\beta}$. Fortunately, however, it will turn out that all coweights which appear in our computation are in the span of the coroots of $\mathbf{G}'$, thus rendering irrelevant the difference between those subgroups of $\mathbf{P}_{\alpha\beta}$ and their image in $\mathbf{L}_{\alpha\beta}$. Indeed, we have:

Remark. Assume that $(\mathbf{Z},\alpha)$ is of type T split, with $\mathbf{D},\mathbf{D}'$ the divisors of smaller rank in $\mathbf{ZP}_\alpha$. While $\check v_D$ is an element of $\mathcal{X}(\mathbf{Z})^*$, it will be convenient in the computations which follow to choose a lift of $\check v_D$ to $\mathrm{Hom}(\mathcal{X}(\mathbf{A}),\mathbb{Q})$ – to be denoted again by $\check v_D$ (and similarly for $\check v_{D'}$). This will not affect the computation of $e^{\check v_D}(\tilde\chi\nu\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi})$ of (5.4) since, as we already remarked, $\tilde\chi\nu\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi}\in A_Z^*$, but it allow us to split it into a product of factors, for instance we will be able to evaluate $e^{\check v_D}(\delta^{\frac12}\delta^{-1}_{(U_{P_\alpha})_\xi})$.

6.5. $\mathbf{SL}_2\setminus\mathbf{SL}_3$. Below is Knop's diagram for the orbits of maximal rank:



We will use the path on the left to compute the coefficient ${}^{\backprime}b^Z_{w_\gamma}$. It is easy to compute that for the orbits D,D' corresponding to the bottom-left reflection of type T we have: $\check v_D = -\check\alpha_1, \check v_D = \check\alpha_1+\check\alpha_2$, $e^{\check v_D}(\delta^{\frac12}\delta^{-1}_{(U_{P_{\alpha_2}})_\xi}) = q^{\frac12}, e^{\check v_{D'}}(\delta^{\frac12}\delta^{-1}_{(U_{P_{\alpha_2}})_\xi}) = 1$. Notice that, because $\check v_D\neq\check v_{D'}$, the spherical root of this variety is necessarily of type T split. Therefore, we get:

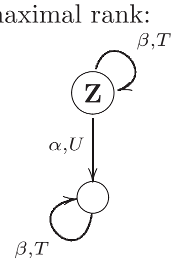
$${}^{\backprime}b^Z_w(\chi) = \frac{1-q^{-1}e^{-\check\alpha_1}}{1-e^{-\check\alpha_1}}(\chi).$$

$$\cdot(-e^{\check\alpha_2})\frac{1-e^{\check\alpha_1}}{1-q^{-1}e^{-\check\alpha_1}}\cdot\frac{1-q^{-1}e^{-\check\alpha_1-\check\alpha_2}}{1-e^{\check\alpha_1+\check\alpha_2}}({}^{w_{\alpha_1}}\chi)\cdot\frac{1-e^{-w\check\alpha}}{1-q^{-1}e^{-w\check\alpha}}(\chi)$$

$$=-e^{\check\alpha_1+\check\alpha_2}\frac{1-q^{-1}e^{-\check\alpha_1}}{1-q^{-1}e^{\check\alpha_1}}\frac{1-q^{-1}e^{-\check\alpha_2}}{1-q^{-1}e^{\check\alpha_2}}(\chi).$$

This agrees with the statement of Proposition 6.4.1.

6.6. $\mathbf{Sp}_2\backslash\mathbf{Sp}_4$. This is a spherical variety for the group $\mathbb{G}_m\times\mathbf{Sp}_4$. (Of course, $\mathbf{Sp}_2=\mathbf{SL}_2$, but we write $\mathbf{Sp}_2$ to emphasize the way it embeds – namely, into the stabilizer of a 2-dimensional symplectic subspace.) Here is Knop's diagram for orbits of maximal rank:



where α is the long root and β is the short one.

We are interested in the spherical root $\alpha+\beta$, since the spherical root β belongs to the type $\mathbf{T}\backslash\mathbf{SL}_2$. One computes that if D,D' are the orbits of co-rank one corresponding to the reflection of type T at the bottom, then $\check v_D=-\check\alpha$, $\check v_{D'}=\check\alpha+\check\beta$, $e^{\check v_D}(\delta^{\frac12}\delta^{-1}_{U_{(P_\beta)_\xi}})=q^{-\frac12}$, $e^{\check v_{D'}}(\delta^{\frac12}\delta^{-1}_{U_{(P_\beta)_\xi}})=q^{\frac12}$. Again, for the same reason as above, the spherical root is necessarily of type T split. Therefore:

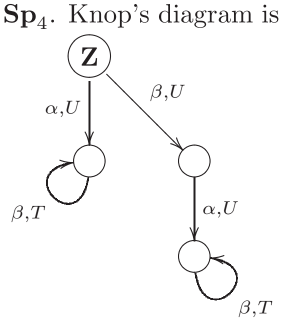
$$`b_w^Z(\chi)=\frac{1-q^{-1}e^{-\check\alpha}}{1-e^{-\check\alpha}}(\chi)\cdot$$

$$\cdot(-e^{\check\beta})\frac{1-e^{\check\alpha}}{1-q^{-1}e^{-\check\alpha}}\cdot\frac{1-q^{-1}e^{-\check\alpha-\check\beta}}{1-e^{\check\alpha+\check\beta}}({}^{w_\alpha}\chi)\cdot\frac{1-e^{-w\check\alpha}}{1-q^{-1}e^{-w\check\alpha}}(\chi)=$$

$$=-e^{2\check\alpha+\check\beta}\frac{1-q^{-1}e^{-\check\alpha}}{1-q^{-1}e^{\check\alpha}}\frac{1-q^{-1}e^{-\check\alpha-\check\beta}}{1-q^{-1}e^{\check\alpha+\check\beta}}(\chi)$$

which agrees with Proposition 6.4.1.

6.7. $\mathbb{G}_a\times\mathbf{Sp}_2\backslash\mathbf{Sp}_4$. Here the subgroup $\mathbb{G}_a$ is the unipotent radical of the "second" $\mathbf{Sp}_2$ inside $\mathbf{Sp}_4$. Knop's diagram is as follows in this case:



where α is the long root and β is the short one.

The left-most path is identical to the one of the previous example, including the same values for $\check{v}_D, e^{\check{v}_D}(\delta^{\frac{1}{2}}\delta^{-1}_{U_{(P_{\alpha_2})_\xi}})$ etc., whence again:

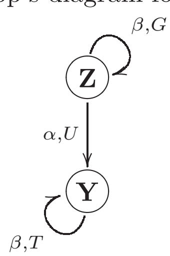
$$`b_w^Z(\chi) = -e^{2\check{\alpha}+\check{\beta}}\frac{1-q^{-1}e^{-\check{\alpha}}}{1-q^{-1}e^{\check{\alpha}}}\frac{1-q^{-1}e^{-\check{\alpha}-\check{\beta}}}{1-q^{-1}e^{\check{\alpha}+\check{\beta}}}(\chi).$$

The path on the right does not correspond to a simple reflection w_γ, because it defines a decomposition of w_γ which is longer than the length of w_γ.

6.8. $\mathbf{Sp}_2 \times \mathbf{Sp}_2 \setminus \mathbf{Sp}_4$**.** Here we have two cases over k, namely:

- the variety $\mathbf{Sp}_2 \times \mathbf{Sp}_2 \setminus \mathbf{Sp}_4$ and
- the variety $(\mathrm{Res}_{E/k}\mathbf{Sp}_2)\setminus \mathbf{Sp}_4$ where E/k is a quadratic field extension and $\mathrm{Res}_{E/k}$ denotes restriction of scalars.

Here is Knop's diagram for orbits of maximal rank:



where α is the long root and β is the short one. The reflection of type T at the bottom is of split type in the first case and of non-split type in the second.

Again, for the divisors of co-rank one at the bottom we have: $\check{v}_D = -\check{\alpha}$, $\check{v}_{D'} = \check{\alpha}+\check{\beta}$, $e^{\check{v}_D}(\delta^{\frac{1}{2}}\delta^{-1}_{U_{(P_\beta)_\xi}}) = q^{-\frac{1}{2}}$, $e^{\check{v}_{D'}}(\delta^{\frac{1}{2}}\delta^{-1}_{U_{(P_\beta)_\xi}}) = q^{\frac{1}{2}}$. Therefore, as before we have in the split case:

$$`b_w^Z(\chi) = -e^{2\check{\alpha}+\check{\beta}}\frac{1-q^{-1}e^{-\check{\alpha}}}{1-q^{-1}e^{\check{\alpha}}}\frac{1-q^{-1}e^{-\check{\alpha}-\check{\beta}}}{1-q^{-1}e^{\check{\alpha}+\check{\beta}}}(\chi).$$

which agrees with Proposition 6.4.1.

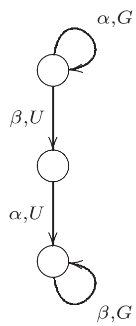
Notice that here $\chi = e^{\frac{\beta}{2}}\chi'$ with $\chi' \in A_Z^*$ and $\check{\alpha} \equiv \check{\alpha}+\check{\beta} = \frac{\check{\gamma}'}{2}$ on $\mathcal{X}(\mathbf{Z})$, therefore the above can also be written:

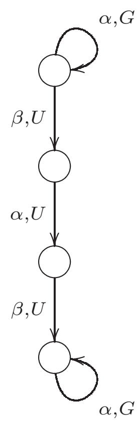
$$`b_w^Z(\chi) = -e^{\check{\gamma}'}\frac{1-q^{-\frac{3}{2}}e^{-\frac{\check{\gamma}'}{2}}}{1-q^{-\frac{3}{2}}e^{\frac{\check{\gamma}'}{2}}}\frac{1-q^{-\frac{1}{2}}e^{-\frac{\check{\gamma}'}{2}}}{1-q^{-\frac{1}{2}}e^{\frac{\check{\gamma}'}{2}}}(\chi').$$

Similarly, in the non-split case we will have:

$$`b_w^Z(\chi) = -e^{\check{\gamma}'}\frac{1-q^{-\frac{3}{2}}e^{-\frac{\check{\gamma}'}{2}}}{1-q^{-\frac{3}{2}}e^{\frac{\check{\gamma}'}{2}}}\frac{1+q^{-\frac{1}{2}}e^{-\frac{\check{\gamma}'}{2}}}{1+q^{-\frac{1}{2}}e^{\frac{\check{\gamma}'}{2}}}(\chi').$$

6.9. **Horospherical varieties.** For the varieties $[\mathbf{P}_\alpha, \mathbf{P}_\alpha]\setminus \mathbf{SL}_3$ or $[\mathbf{P}_\alpha, \mathbf{P}_\alpha]\setminus \mathbf{Sp}_4$ of Proposition 6.3.1, the graphs look as follows:



and


, respectively.

The validity of (6.10) at every intermediate node is easily verified.

6.10. **Spherical varieties of rank at most two.** We now come to the proof of Proposition 6.3.2. The starting point is classification of wonderful varieties of rank one by Akhiezer [Ak83] and of rank two by Wasserman [Was96].

6.10.1. **Theorem.** *Below is a complete list of spherical varieties $\mathbf{X}' = \mathbf{H}'\backslash\mathbf{G}'$ over $\bar{k}$ with the following properties:*

- *$\mathbf{G}'$ is semisimple, simply connected, and $\mathbf{H}'$ is equal to the connected component of its normalizer.*
- *There is a spherical root γ whose support is the set of simple roots of $\mathbf{G}'$.*
- *There are no spherical roots of type N.*

(For simplicity and since everything is over the algebraic closure, we do not use boldface letters in what follows. We also write k^n to denote an n-dimensional vector space.)

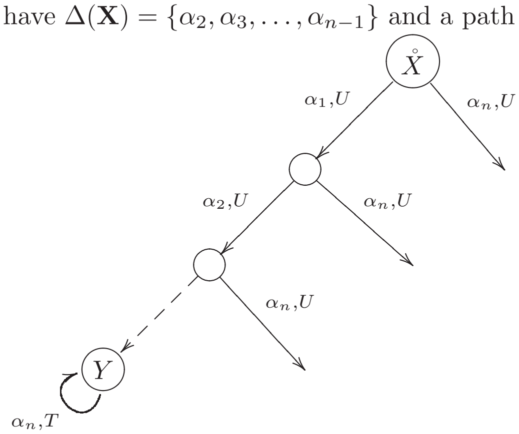
$\mathbf{G}'$	$\mathbf{H}'$
Rank one:	
SL_{n+1}	GL_n
$\mathrm{SL}_2 \times \mathrm{SL}_2$	$\mathrm{SL}_2^{\mathrm{diag}}$
SL_4	Sp_4
Spin_{2n+1}	Spin_{2n}
Spin_{2n+1}	$(\mathrm{SL}_n \times \mathbb{G}_m) \ltimes \wedge^2 k^n$
Spin_7	G_2
Sp_{2n}	$\mathrm{Sp}_2 \times \mathrm{Sp}_{2n-2}$
Sp_{2n}	$(\mathbb{G}_m \ltimes k) \times \mathrm{Sp}_{2n-2}$
Spin_{2n}	Spin_{2n-1}
F_4	Spin_9
G_2	SL_3
G_2	$(\mathbb{G}_m \times \mathrm{SL}_2) \ltimes (k^1 \oplus k^2)$
Rank two:	
Spin_9	Spin_7
Sp_{2n}	$\mathbb{G}_m \times \mathrm{Sp}_{2n-2}$
G_2	$(\mathbb{G}_m \times \mathrm{SL}_2) \ltimes k^2$.

(In the case of $G' = \mathrm{Spin}_{2n}$ and $H' = \mathrm{Spin}_{2n-1}$, $n = 4$, this includes the "non-obvious" embedding of Spin_7 obtained from the "obvious" one by applying the outer automorphism of Spin_8.)

Proof. If we add the extra condition "the rank of $\mathbf{X}'$ is at most two", then it follows by inspection of the tables of [Was96]. As we explain in [SV], for *any* spherical variety $\mathbf{X}$ and *any* spherical root γ of it, the spherical variety $\mathbf{X}_\gamma$ is of rank at most two. Therefore, the condition on the rank of $\mathbf{X}'$ being at most two is vacuous. □

Let γ denote the spherical root in each of the above whose support is the set of all simple roots of $\mathbf{G}'$. To prove Proposition 6.3.2, it suffices to check that for each of the varieties above (where $\mathbf{G}'$ is a classical group) and for every path corresponding to γ there is a path leading to γ with the same first edge which is glued out of the spherical varieties of Proposition 6.3.1. We notice that the spherical roots for the examples below can be read off from the tables of [Was96], and that for a quasi-affine spherical variety $\mathbf{X}$ the set $\Delta(\mathbf{X})$ consists precisely of the simple roots of $\mathbf{G}$ which are orthogonal to $\mathcal{X}(\mathbf{X})$ [Kn94, Lemma 3.1]. There is nothing to check for $\mathbb{G}_m \backslash \mathbf{SL}_2$ and $\mathbf{SL}_2 \backslash \mathbf{SL}_2 \times \mathbf{SL}_2$.

6.11. **$\mathbf{GL}_n \backslash \mathbf{SL}_{n+1}$, $n \geq 2$.** The spherical root here is $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, we have $\Delta(\mathbf{X}) = \{\alpha_2, \alpha_3, \ldots, \alpha_{n-1}\}$ and a path corresponding to it is the following:

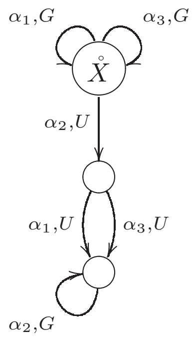


where the arrows not shown are reflections of type G.

Clearly, this path is glued from $\mathbf{SL}_2 \backslash \mathbf{SL}_3$ (at the bottom) and $\mathbf{P}_\alpha \backslash \mathbf{SL}_3$. There is also a similar path starting with α_n. We mention that here we have:

$$B_{w_\gamma}(\chi) = -e^{\check{\gamma}} \frac{(1 - q^{-\frac{n}{2}} e^{-\check{\alpha}_1})(1 - q^{-\frac{n}{2}} e^{-\check{\alpha}_n})}{(1 - q^{-\frac{n}{2}} e^{\check{\alpha}_1})(1 - q^{-\frac{n}{2}} e^{\check{\alpha}_n})}(\chi). \tag{6.11}$$

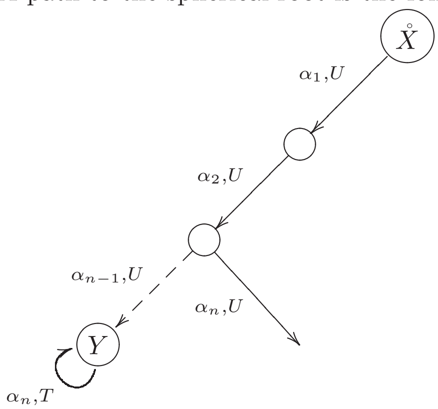
6.12. **$\mathbf{Sp}_4 \backslash \mathbf{SL}_4$.** Here $\Delta(\mathbf{X}) = \{\alpha_1, \alpha_3\}$ and the spherical root is $\gamma = \alpha_1 + 2\alpha_2 + \alpha_3$. Knop's diagram is as follows:



Therefore, this path is glued from varieties of the form $\mathbf{PGL}_2 \backslash \mathbf{PGL}_2 \times \mathbf{PGL}_2$ and $\mathbf{P}_\alpha \backslash \mathbf{SL}_3$. Here we have:

$$B_{w_\gamma}(\chi) = -e^{\check{\gamma}} \frac{1 - q^{-2} e^{-\check{\gamma}}}{1 - q^{-2} e^{\check{\gamma}}}(\chi). \tag{6.12}$$

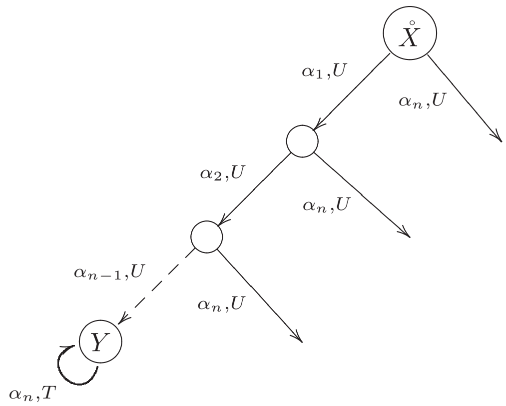
6.13. $\mathbf{Spin}_{2n} \backslash \mathbf{Spin}_{2n+1}$. We denote by $\alpha_1, \dots, \alpha_{n-1}$ the long roots and by α_n the short one. The spherical root is $\gamma = \alpha_1 + \alpha_2 + \dots + \alpha_n$, and $\Delta(\mathbf{X}) = \{\alpha_2, \dots, \alpha_n\}$. A path to the spherical root is the following:



It is glued from $\mathbf{Sp}_2 \times \mathbf{Sp}_2 \backslash \mathbf{Sp}_4$ (at the bottom) and $\mathbf{P}_\alpha \backslash \mathbf{SL}_3$. We have:

$$B_{w_\gamma}(\chi) = -e^{\check{\gamma}} \frac{(1 - q^{-n} e^{-\frac{\check{\gamma}}{2}})(1 - q^{-\frac{1}{2}} e^{-\frac{\check{\gamma}}{2}})}{(1 - q^{-n} e^{\frac{\check{\gamma}}{2}})(1 - q^{-\frac{1}{2}} e^{\frac{\check{\gamma}}{2}})}(\chi). \tag{6.13}$$

6.14. $(\mathbf{SL}_n \times \mathbb{G}_m) \ltimes \wedge^2 k^n \backslash \mathbf{Spin}_{2n+1}$. With notation as above, the spherical root is the same as above (namely, $\gamma = \alpha_1 + \dots + \alpha_n$) and $\Delta(\mathbf{X}) = \{\alpha_2, \dots, \alpha_{n-1}\}$. A path to the spherical root is the following:

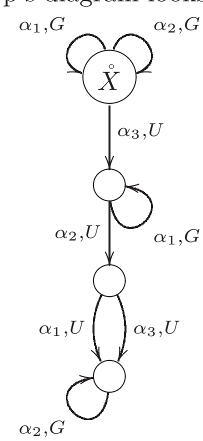


It is glued from $(\mathbb{G}_m \ltimes \mathbb{G}_a) \times \mathbf{Sp}_2 \backslash \mathbf{Sp}_4$ (at the bottom) and $\mathbf{P}_\alpha \backslash \mathbf{SL}_3$. The edge α_n on the top-right cannot lead to w_γ since α_n is orthogonal to γ and hence the length of any such path would be larger than the length of w_γ.

Here we have:

$$B_{w_\gamma}(\chi) = -e^{\check{\gamma}} \frac{(1-q^{-\frac{n}{2}} e^{-\check{\alpha}_1})(1-q^{-\frac{n}{2}} e^{-\check{\alpha}_1-\check{\alpha}_n})}{(1-q^{-\frac{n}{2}} e^{\check{\alpha}_1})(1-q^{-\frac{n}{2}} e^{\check{\alpha}_1+\check{\alpha}_n})}(\chi). \tag{6.14}$$

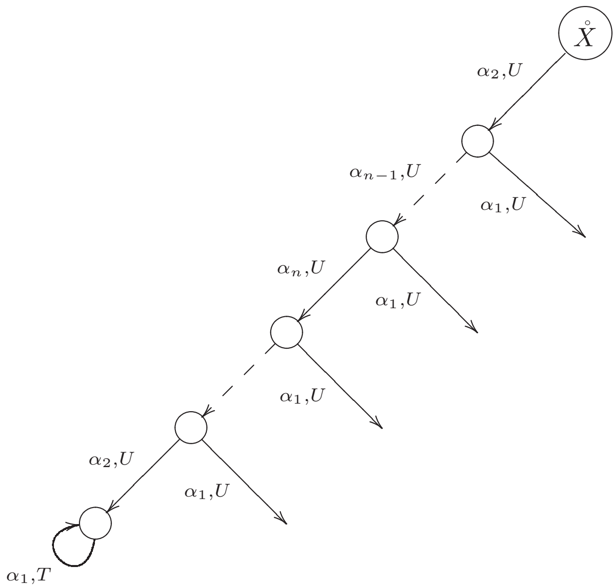
6.15. $\mathbf{G}_2 \backslash \mathbf{Spin_7}$**.** The spherical root is $\gamma = \alpha_1 + 2\alpha_2 + 3\alpha_3$, and $\Delta(\mathbf{X}) = \{\alpha_1, \alpha_2\}$. Knop's diagram looks as follows:



It is glued from the varieties $\mathbf{P}_\alpha \backslash \mathbf{Sp}_4$ (where α denotes the long root), $\mathbf{P}_\alpha \backslash \mathbf{SL}_3$ and $\mathbf{PGL}_2 \backslash \mathbf{PGL}_2 \times \mathbf{PGL}_2$. We have:

$$B_{w_\gamma}(\chi) = -e^{\check{\gamma}} \frac{1-q^{-3}e^{-\check{\gamma}}}{1-q^{-3}e^{\check{\gamma}}}(\chi). \tag{6.15}$$

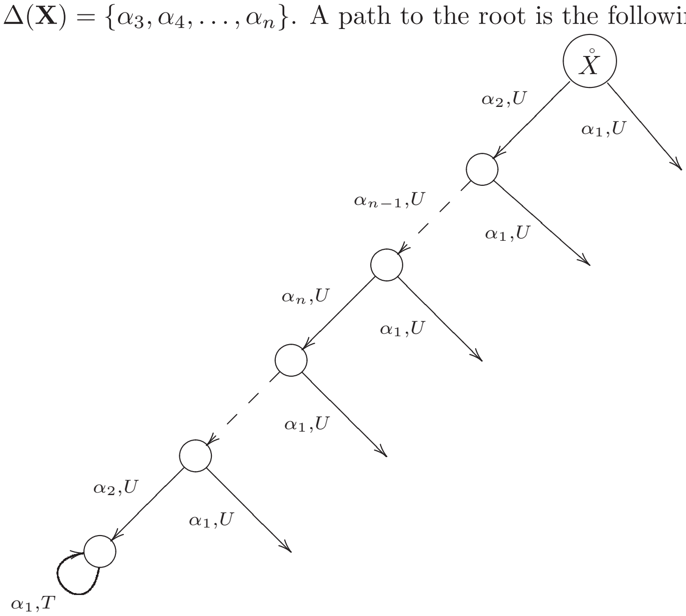
6.16. $\mathbf{Sp}_2 \times \mathbf{Sp}_{2n-2} \backslash \mathbf{Sp}_{2n}$**.** Denote by $\alpha_1, \dots, \alpha_{n-1}$ the short roots and by α_n the long one. Then the spherical root is $\gamma = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-1} + \alpha_n$, and $\Delta(\mathbf{X}) = \{\alpha_1, \alpha_3, \alpha_4, \dots, \alpha_n\}$. A path for the spherical root is the following:



It is glued from $\mathbf{GL}_2 \backslash \mathbf{SL}_3$ (at the bottom), $\mathbf{P}_\alpha \backslash \mathbf{Sp}_4$ (where α is the long root) and $\mathbf{P}_\alpha \backslash \mathbf{SL}_3$, unless $n = 2$, in which case it coincides with $\mathbf{Sp}_2 \times \mathbf{Sp}_2 \backslash \mathbf{Sp}_4$. We have here:

$$B_{w_\gamma}(\chi) = -e^{\check{\gamma}} \frac{(1 - q^{-n+\frac{3}{2}} e^{-\frac{\check{\gamma}}{2}})(1 - q^{-n+\frac{1}{2}} e^{-\frac{\check{\gamma}}{2}})}{(1 - q^{-n+\frac{3}{2}} e^{\frac{\check{\gamma}}{2}})(1 - q^{-n+\frac{1}{2}} e^{\frac{\check{\gamma}}{2}})}(\chi). \tag{6.16}$$

6.17. $(\mathbb{G}_m \ltimes k) \times \mathbf{Sp}_{2n-2} \backslash \mathbf{Sp}_{2n}$**.** The spherical root is the same as above, and $\Delta(\mathbf{X}) = \{\alpha_3, \alpha_4, \ldots, \alpha_n\}$. A path to the root is the following:

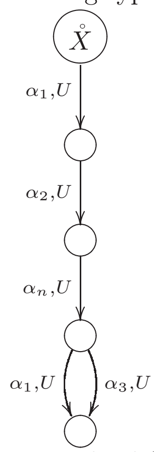


which, again, is glued from $\mathbf{GL}_2 \backslash \mathbf{SL}_3$ (at the bottom), $\mathbf{P}_\alpha \backslash \mathbf{Sp}_4$ (where α is the long root) and $\mathbf{P}_\alpha \backslash \mathbf{SL}_3$ (unless $n = 2$, a case which was treated before). The edge labelled α_1 on the top right cannot lead to w_γ, since α_1 is orthogonal to γ.

Here we have:

$$B_{w_\gamma}(\chi) = -e^{\check\gamma}\frac{(1-q^{-n+1}e^{-\check\alpha_2})(1-q^{-n+1}e^{-\check\alpha_1-\check\alpha_2})}{(1-q^{-n+1}e^{\check\alpha_2})(1-q^{-n+1}e^{\check\alpha_1+\check\alpha_2})}(\chi). \tag{6.17}$$

6.18. $\mathbf{Spin}_{2n-1}\backslash\mathbf{Spin}_{2n}$**.** Denote the simple roots by $\alpha_1,\alpha_2,\dots,\alpha_n$, where α_{n-2} is the root neighboring with three others ($\alpha_{n-3},\alpha_{n-1}$ and α_n). Assume $n\ge 3$, the case $n=3$ having been treated in §6.11. The spherical root is $\gamma = 2\alpha_1+\dots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n$, and $\Delta(\mathbf{X}) = \{\alpha_2,\dots,\alpha_n\}$. Knop's graph looks as follows (omitting type G reflections):

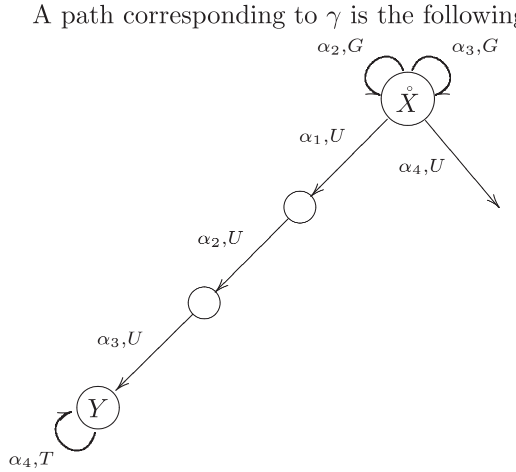


It is glued from $\mathbf{PGL}_2\backslash\mathbf{PGL}_2\times\mathbf{PGL}_2$ and $\mathbf{P}_\alpha\backslash\mathbf{SL}_3$. We have:

$$B_{w_\gamma}(\chi) = -e^{\check\gamma}\frac{1-q^{-n+1}e^{-\check\gamma}}{1-q^{-n+1}e^{\check\gamma}}(\chi). \tag{6.18}$$

6.19. $\mathbf{Spin}_7\backslash\mathbf{Spin}_9$**.** There are 2 spherical roots, $\gamma = \alpha_1+\alpha_2+\alpha_3+\alpha_4$ and $\alpha_2+2\alpha_3+3\alpha_4$ but, as in §6.6, we are interested only in γ. Here $\Delta(\mathbf{X}) = \{\alpha_2,\alpha_3\}$.

A path corresponding to γ is the following:

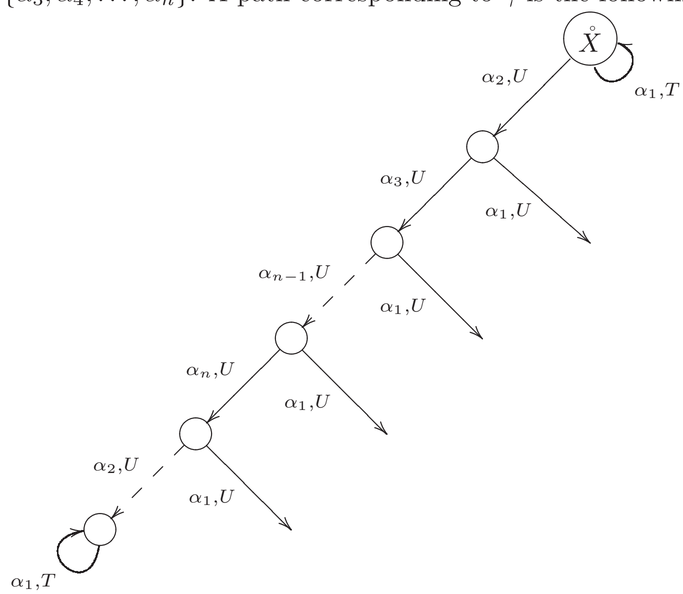


where we have only drawn the arrows of interest.

It is glued from $(\mathbb{G}_m\ltimes\mathbb{G}_a)\times\mathbf{Sp}_2\backslash\mathbf{Sp}_4$ (at the bottom) and $\mathbf{P}_\alpha\backslash\mathbf{SL}_3$. The arrow labelled α_4 on the top cannot lead to w_γ since α_4 is orthogonal to w_γ. We have:

$$B_{w_\gamma}(\chi) = -e^{\check\gamma}\frac{(1-q^{-2}e^{-\check\alpha_1})(1-q^{-2}e^{-\check\alpha_1-\check\alpha_4})}{(1-q^{-2}e^{\check\alpha_1})(1-q^{-2}e^{\check\alpha_1+\check\alpha_4})}(\chi). \tag{6.19}$$

6.20. $\mathbb{G}_m \times \mathbf{Sp}_{2n-2} \backslash \mathbf{Sp}_{2n}$. There are 2 spherical roots, α_1 and $\gamma = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$ but, as in §6.6, we are interested only in γ. We have $\Delta(\mathbf{X}) = \{\alpha_3, \alpha_4, \dots, \alpha_n\}$. A path corresponding to γ is the following:



which is glued from $\mathbf{SL}_2 \backslash \mathbf{SL}_3$ (at the bottom), $\mathbf{P}_\alpha \backslash \mathbf{Sp}_4$ (where α is the long root) and $\mathbf{P}_\alpha \backslash \mathbf{SL}_3$ (unless $n = 2$, which was treated before). We have:

$$B_{w_\gamma}(\chi) = -e^{\check{\gamma}} \frac{(1 - q^{-n+1} e^{-\check{\alpha}_2})(1 - q^{-n+1} e^{-\check{\alpha}_1 - \check{\alpha}_2})}{(1 - q^{-n+1} e^{\check{\alpha}_2})(1 - q^{-n+1} e^{\check{\alpha}_1 + \check{\alpha}_2})}(\chi). \tag{6.20}$$

7. The formula

The goal of this section is to develop a more useful and explicit formula, in which the cocycles of our previous formula have been substituted by simpler expressions, and where a certain quotient of L-values plays a distinguished role. In order to do this, we need to make a combinatorial assumption on the colors of a spherical variety, which is very easy to check in each particular case, but we do not know how to prove. This assumption is certainly not true in all cases, but we expect it to be true in the case of affine homogeneous spherical varieties.

7.1. **Relevant and virtual colors.** Assume that all spherical roots of $\mathbf{X}$ satisfy Statement 6.2.1. We call a color of $\mathbf{X}$ *relevant* if it lies at the beginning of a path corresponding to w_γ, for γ a simple spherical root.

7.1.1. **Lemma.** *If* $\mathbf{X}$ *is affine, then all colors are relevant.*

Proof. We give just a sketch of the proof, since it will not be used later: Let $\mathbf{D}$ be an irrelevant color with valuation $\check{\theta}$, and let α be a simple root raising it to $\mathring{\mathbf{X}}$. If $(\mathring{\mathbf{X}}, \alpha)$ were of type T then $\mathbf{D}$ would be relevant, so $(\mathring{\mathbf{X}}, \alpha)$ is of type U, and hence $\check{\theta} = \check{\alpha}$. If $\alpha \in \operatorname{supp}(\gamma)$ for some spherical root γ, then $\mathbf{X}_\gamma$ is of rank at most two and one sees by a case-by-case analysis of spherical varieties of rank at most two that $\check{\alpha} \perp \gamma$. On the other hand, if $\alpha \notin \operatorname{supp}(\gamma)$ then $\langle \check{\alpha}, \gamma \rangle \leq 0$. Hence $\langle \check{\alpha}, \gamma \rangle \leq 0$ for all $\gamma \in \Delta_X$, which means that $\check{\theta} = \check{\alpha} \in \mathcal{V}$. But if $\mathbf{X}$ is affine homogeneous then [Kn91,

§6] the cone generated by valuations of colors is separated from $\mathcal{V}$ by a hyperplane, a contradiction. □

We now make some remarks assuming the validity of Statement 6.2.1:

- A color $\mathbf{D}$ with corresponding valuation $\check\theta$ is in the beginning of a path leading to a simple spherical reflection w_γ if and only if $\langle\check\theta,\gamma\rangle > 0$. In this case, we will say that $\mathbf{D}$ *belongs* to the spherical root γ.
- If a color belongs to a spherical root of type G, type T non-split, or type (U,ψ), then it belongs only to that spherical root. Therefore, one can define the *type* of a relevant color as being one among the following: G, T split, T non-split or (U,ψ).

We now define the set of *virtual weighted colors* as follows: It consists of triples $(\check\theta,\sigma,r)$ where $\check\theta$ is an element of $\mathcal{X}(\mathbf{X})^*$, $\sigma=\pm$ is a sign and $r\in\frac{1}{2}\mathbb{Z}$; for every relevant color $\mathbf{D}$ with valuation $\check\theta$, the triple $(\check\theta,+,\langle\check\theta,\rho_{P(X)}\rangle)$ is a weighted color; if γ is a simple spherical root of type T split and the color $\mathbf{D}$ (with valuation $\check\theta$) belongs to γ then the triple $(-^{w_\gamma}\check\theta,+,\langle\check\rho,\gamma\rangle-\langle\check\theta,\rho_{P(X)}\rangle)$ is also a virtual weighted color – in case it coincides with $(\check\theta,+,\langle\check\theta,\rho_{P(X)}\rangle)$ it is counted with multiplicity two; finally, if γ is a simple spherical root of type T non-split and the color $\mathbf{D}$ (with valuation $\check\theta$) belongs to γ then the triple $(-^{w_\gamma}\check\theta,-,\langle\check\rho,\gamma\rangle-\langle\check\theta,\rho_{P(X)}\rangle)$ is also a virtual weighted color. No other triples are virtual weighted colors.

We clarify the following fact on multiplicities: To every simple spherical root $\check\gamma$ of type T there are two "belonging" virtual weighted colors $(\check\theta,\sigma,r)$, characterized by the fact that $\langle\check\theta,\gamma\rangle>0$. If in case T non-split there are two "real" belonging colors (necessarily with the same valuation $\check\theta=\frac{\check\gamma}{2}$), then one is taken as a virtual weighted color with positive sign and the other is taken with negative sign. In case T split there may be two identical virtual weighted colors, or there may be virtual weighted colors that belong to more than one simple spherical root. Not all virtual weighted colors come from real colors, see for instance the examples of §6.6 and §6.8.

The group W_X acts on triples $(\check\theta,\sigma,r)$ as above by acting on the first component. We let Θ denote the minimal set of such triples such that:

- Θ contains all virtual weighted colors.
- If $(\check\theta,\sigma,r)\in\Theta$ then $(-\check\theta,\sigma,r)\in\Theta$ (and with the same multiplicity).

By abuse of notation, we will sometimes write $\check\theta\in\Theta$ and mean a triple as above, in which case we will denote by $\sigma_{\check\theta}$, $r_{\check\theta}$ the corresponding sign and constant. We denote by Θ^+ the set of $\check\theta\in\Theta$ such that $\langle\check\theta,\eta\rangle\ge 0$ for every $\eta\in\mathcal{X}(\mathbf{X})$ which appears in $k[\mathbf{X}]^{(\mathbf{B})}$. Equivalently, since the regular $\mathbf{B}$-eigenfunctions are precisely those rational $\mathbf{B}$-eigenfunctions which don't blow up on any of the colors, Θ^+ consists of all $\check\theta\in\Theta$ such that $\check\theta$ lies in the cone generated by the valuations defined by all colors. (This cone is strictly convex since $\mathbf{X}$ is assumed quasi-affine.) We also write $\check\theta>0$ if $\check\theta\in\Theta^+$, and $\check\theta<0$ if $-\check\theta\in\Theta^+$.

Our main assumption regarding the set Θ is the following:

7.1.2. **Statement.** *For every $(\check\theta,\sigma,r)\in\Theta$ we either have $\check\theta>0$ or $-\check\theta<0$. For every $\gamma\in\Delta_X$ the set $\{(\check\theta,\sigma,r)\in\Theta^+|w_\gamma\check\theta<0\}$ consists precisely of the virtual weighted colors belonging to γ.*

This assumption is certainly not true for every spherical variety. However, I conjecture that it is true if $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ is (homogeneous and) affine, i.e. $\mathbf{H}$ is reductive. The benefit of this stament is that it is very easy to check in each particular case; its drawback is that it gives no insight into what it might mean. A geometric understanding of this stament may provide a better understanding of the nature of the L-functions which are about to appear. At the end of this section we will discuss an analytic criterion which would – via a representation-theoretic argument – imply this assumption, and which may be a way to understand it. (Although at this point it seems at least equally hard to prove.) But first we will see what this assumption implies.

7.2. **The formula.** The final form of our formula is the following:

7.2.1. **Theorem.** *Assume that all simple spherical roots of* $\mathbf{X}$ *satisfy Statement 6.2.1 and that the set* Θ *satisfies Statement 7.1.2. Then:*

$$\Omega_{\delta_{(X)}^{\frac{1}{2}}\tilde{\chi}}(x_0) = \omega(\tilde{\chi})\beta(\tilde{\chi}) \tag{7.1}$$

where

$$\beta(\tilde{\chi}) := \frac{\prod_{\check{\gamma}\in\check{\Phi}_X^+}(1-e^{\check{\gamma}})}{\prod_{\check{\theta}\in\Theta^+}(1-\sigma_{\check{\theta}}q^{-r_{\check{\theta}}}e^{\check{\theta}})}(\tilde{\chi}), \tag{7.2}$$

and $\omega \in \mathbb{C}[A_X^*]^{W_X}$*. If* $\mathbf{X}$ *is affine, then* ω *is a constant* c *equal to:*

$$c = \beta(\delta_{P(X)}^{\frac{1}{2}})^{-1}. \tag{7.3}$$

In either case:

$$\frac{\Omega_{\delta_{(X)}^{\frac{1}{2}}\tilde{\chi}}(x_{\check{\lambda}})}{\beta(\tilde{\chi})} = \delta_{P(X)}^{-\frac{1}{2}}(x_{\check{\lambda}})\prod_{\Theta^+}(1-\sigma_{\check{\theta}}q^{-r_{\check{\theta}}}T_{\check{\theta}})s_{\check{\lambda}}(\tilde{\chi}) \tag{7.4}$$

where $s_{\check{\lambda}} = \frac{\sum_{W_X}(-1)^w e^{\check{\rho}-w\check{\rho}+w\check{\lambda}}}{\prod_{\check{\gamma}>0}(1-e^{\check{\gamma}})}$ *is the Schur polynomial* indexed by lowest weight *(that is, if* $\check{\lambda}$ *is anti-dominant then* $s_{\check{\lambda}}$ *is the character of the irreducible representation of* $\check{G}_X$ *with lowest weight* $\check{\lambda}$*) and* $T_{\check{\theta}}$ *denotes the formal operator:* $T_{\check{\theta}}s_{\check{\lambda}} = s_{\check{\theta}+\check{\lambda}}$*.*

Remark. In the case of a pair $(X, \mathcal{L}_\Psi)$, where X is parabolically induced from an *affine* spherical variety and Ψ is a generic character, then it should be considered that we are in the same case as that of an "affine variety". The statements above pertaining to affine varieties hold, except for the one about the value of the constant c, due to failure of Lemma 4.2.3.

The proof of the theorem will be completely combinatorial. The only geometric information used is the following fact (cf. [Kn91, §6]):

> The spherical variety $\mathbf{X} = \mathbf{H}\backslash\mathbf{G}$ is affine (equivalently, $\mathbf{H}$ is reductive) if and only if $\mathcal{V}$ and $\{\check{v}_D|\mathbf{D} \text{ a color}\}$ are separated by a hyperplane. It is quasi-affine if and only if $\{\check{v}_D|\mathbf{D} \text{ a color}\}$ spans a strictly convex cone and does not contain zero.

Proof. The first step is to show that $B_w(\tilde\chi) = \frac{\beta(\tilde\chi)}{\beta({}^w\tilde\chi)}$. Since the B_w satisfy cocycle relations, it is enough to show that for $w = w_\gamma$, where $\gamma \in \Delta_X$. We compute:

$$\frac{\beta(\tilde\chi)}{\beta({}^{w_\gamma}\tilde\chi)} = -e^{\check\gamma}(\tilde\chi) \cdot \prod_{\check\theta>0, w_\gamma\check\theta<0} \frac{1-\sigma_{\check\theta}q^{-r_{\check\theta}}e^{-\check\theta}}{1-\sigma_{\check\theta}q^{-r_{\check\theta}}e^{\check\theta}}.$$

By Theorem 7.1.2 and Statement 6.2.1, for $w = w_\gamma$ this is equal to $B_{w_\gamma}(\tilde\chi)$. Hence,

$$\frac{\Omega_{\delta_{(X)}^{\frac12}\tilde\chi}(x_{\check\lambda})}{\beta(\tilde\chi)} = \delta_{P(X)}^{-\frac12}(x_{\check\lambda}) \sum_{W_X} \frac{1}{\beta({}^w\tilde\chi)}\, {}^w\chi(x_{\check\lambda}).$$

This proves (7.4), which in particular implies that $\frac{\Omega_{\delta_{(X)}^{\frac12}\tilde\chi}(x_{\check\lambda})}{\beta(\tilde\chi)}$ is *regular* in $\tilde\chi$, for every $\check\lambda$. In particular, for $\check\lambda = 0$ we have $\frac{\Omega_{\delta_{(X)}^{\frac12}\tilde\chi}(x_0)}{\beta(\tilde\chi)} = \omega(\tilde\chi)$ where ω is a W_X-invariant *regular* function of $\tilde\chi$. Our proof will be complete if we show that in the affine case ω is a *constant*; the statement about the precise value of c follows from the fact that $\Omega_{\delta^{\frac12}} = 1$ (Lemma 4.2.3).

We define two partial orders on the set of weights on A_X^*: We will write that $\check\lambda \succ_1 \check\mu$ if $\check\lambda - \check\mu$ is in the non-negative span $\mathcal{R}$ of $\check\Phi_X^+$; and that $\check\lambda \succ_2 \check\mu$ if $\check\lambda - \check\mu$ is in the non-negative span $\mathcal{T}$ of the valuations induced by all colors (i.e. the cone dual to the cone of characters of $\mathbf{B}$ on $k[\mathbf{X}]^{(\mathbf{B})}$. Since $\mathcal{T} \supset \mathcal{R}$ (see the proof of Theorem 7.1.2), the second order is weaker than the first. Now we have:

7.2.2. **Lemma.** *If ω is a non-constant, (W_X-)symmetric polynomial on A_X^* and $\check\lambda$ is a minimal weight appearing with non-zero coefficient in ω, then $\check\lambda$ is also a minimal weight, appearing with the same coefficient, in $\omega \cdot \prod_{\check\gamma\in\check\Phi_X^+}(1-e^{\check\gamma})$.*

The validity of this lemma is obvious, since "minimal weight" means minimal for the $\succ_1$ ordering and all weights of $\prod_{\check\gamma\in\check\Phi_X^+}(1-e^{\check\gamma})$ are $\succ_1 0$, with strict inequality except for the summand which is equal to the constant 1.

Given this lemma and (7.4), it suffices to prove that $\sum_{W_X}(-1)^w e^{\check\rho - w\check\rho + w\check\lambda}$ does not contain any anti-dominant weights when $\check\lambda = \sum_{\check\theta\in I}\check\theta$ and I is any subset of Θ^+. Here we use the fact that $\mathcal{V}$ and $\{\check v_D | D \text{ a color}\}$ are separated by a hyperplane. By Proposition 7.1.2 and Statement 6.2.1, this means that:

$\mathcal{V}$ and Θ^+ are separated by a hyperplane

or, equivalently, that $\mathcal{V}$ and $\mathcal{T}$ intersect trivially. Since we want to prove that for every $w \in W_X, I \subset \Theta^+$ the weight $\check\rho - w\check\rho + w\sum_{(\check\theta,r)\in I}\check\theta$ is not anti-dominant, it suffices to show that:

$$\check\rho - w\check\rho + w \sum_{(\check\theta,r)\in I} \check\theta \succ_2 0. \tag{7.5}$$

For simplicity of notation, in what follows, we denote by $|I|$ the sum $\sum_{(\check\theta,r)\in I}\check\theta$, for any $I \subset \Theta$.

Given that $|\Theta^+| - w|\Theta^+| = \check\rho - w\check\rho$, we can write (7.5) as:

$$|\Theta^+| - w(|\Theta^+| - |I|) \succ_2 0.$$

Let $\Theta_1 = w\Theta^+ \cap \Theta^+$ and $\Theta_2 = w\Theta^+ \smallsetminus \Theta_1 \subset -\Theta^+$. Let $I_1 = w(\Theta^+ \smallsetminus I) \cap \Theta_1$ and $I_2 = w(\Theta^+ \smallsetminus I) \cap \Theta_2$. Then (7.5) can be written:

$$|\Theta^+| - |I_1| + |-I_2| \succ_2 0 \iff |\Theta^+ \smallsetminus I_1| + |-I_2| \succ_2 0.$$

Since both $\Theta^+ \smallsetminus I_1$ and $-I_2$ belong to Θ^+, this holds! $\square$

Remark. With minor additions, this proof actually shows that:

$$\prod_{\Theta^+}(1-\sigma_{\check\theta} q^{-r_{\check\theta}} T_{\check\theta}) s_{\check\lambda}(\tilde\chi) - \sum_{W_X} e^{w\check\lambda} \succ_2 \check\lambda,$$

where, for a symmetric polynomial p, the expression $p \succ_2 \check\lambda$ means that all weights of p are $\succ_2 \check\lambda$; equivalently, since $\succ_2$ is weaker than $\succ_1$, that all anti-dominant weights of p are $\succ_2 \check\lambda$.

The theorem leads us to the following definition:

7.2.3. **Definition.** We denote by $L_X^{\frac12}$ the function

$$\tilde\chi \mapsto c\beta(\tilde\chi) = c\frac{\prod_{\check\gamma\in\check\Phi_X^+}(1-e^{\check\gamma})}{\prod_{\check\theta\in\Theta^+}(1-\sigma_{\check\theta} q^{-r_{\check\theta}} e^{\check\theta})}(\tilde\chi)$$

on A_X^*.

We denote by L_X and call "the L-function of X" the function

$$L_X^{\frac12}(\tilde\chi)L_X^{\frac12}(\tilde\chi^{-1}) = c^2\frac{\prod_{\check\gamma\in\check\Phi_X}(1-e^{\check\gamma})}{\prod_{\check\theta\in\Theta}(1-\sigma_{\check\theta} q^{-r_{\check\theta}} e^{\check\theta})},$$

which is W_X-invariant on A_X^* and hence can be thought of as a conjugation-invariant function on $\check G_X$.

Remark. The importance of this definition lies in the (mostly conjectural) relationship between period integrals of automorphic forms and the value of $\Omega_{\delta^{\frac12}_{(X)}\tilde\chi}$ in the affine, multiplicity-free case, see section 10.

7.2.4. *Example.* Let $\mathbf{X} = \mathbf{PGL}_2 \backslash (\mathbf{PGL}_2)^3$, the variety of Example 5.6.3. Here the valuations induced by the colors are $\frac{\check\alpha_1+\check\alpha_2-\check\alpha_3}{2}$, $\frac{\check\alpha_1-\check\alpha_2+\check\alpha_3}{2}$, $\frac{-\check\alpha_1+\check\alpha_2+\check\alpha_3}{2}$ and the set Θ^+ contains those and also the co-weight $\frac{\check\alpha_1+\check\alpha_2+\check\alpha_3}{2}$ (for simplicity, since $\sigma_{\check\theta} = +$ and $r_{\check\theta} = \frac12$ for all $\check\theta$, we only write the value of $\check\theta$). Clearly, $\Theta = \Theta^+ + (-\Theta^+)$ is $W = W_X$-invariant, and therefore the final formula of Example 5.6.3 holds. In this example, L_X is up to zeta-factors equal to the the quotient of the tensor product L-function at $\frac12$ by the adjoint L-function at 0.

7.2.5. *Example.* This example is related to the period integral proposed by Gross and Prasad, cf. [GP92]. Let $\mathbf{X} = \mathbf{SO}_n \backslash \mathbf{SO}_n \times \mathbf{SO}_{n+1}$ (the stabilizer embedded diagonally in the product of the two groups). Assume all groups are split. One can check that in this case $\check G_X = \check G$, and that the set Θ^+ satisfies Statement 7.1.2 and is equal to the set of all non-trivial weights $\check\theta$ of the tensor-product representation of $\check G_X$ such that $\langle\check\theta,\rho\rangle > 0$ (where ρ denotes, as usual, the half-sum of positive roots of $\mathbf{G}$, and all $\sigma_{\check\theta} = +$ and $r_{\check\theta} = \frac12$).[11] Hence, the corresponding L-value L_X

[11]We remark that the valuations associated to colors of type T are usually very easy to compute as follows: One computes the valuation associated to a color in $\mathring{\mathbf{X}}\mathbf{P}_\alpha$ by computing the stabilizer in $\mathbf{P}_\alpha$ of a point and using the fact that $\langle\check v_D,\alpha\rangle = 1$. Then one uses the fact that for such a divisor $\mathbf{D}$ we have $\mathbf{D} \in \mathring{\mathbf{X}}\mathbf{P}_\beta$ if and only if $\langle\check v_D,\beta\rangle = 1$ and that if $\mathbf{D},\mathbf{D}'$ are the divisors in $\mathring{\mathbf{X}}\mathbf{P}_\beta$

is, up to zeta factors, equal to:

$$\frac{L(\pi_1 \otimes \pi_2, \frac{1}{2})}{L(\pi_1, \mathrm{Ad}, 0) L(\pi_2, \mathrm{Ad}, 0)}$$

where we have decomposed an unramified representation π of G as $\pi_1 \otimes \pi_2$ according to the decomposition $\mathbf{G} = \mathbf{SO}_n \times \mathbf{SO}_{n+1}$. In this case our calculation is similar to that performed (in greater generality) by Ichino and Ikeda [II].

7.2.6. *Example.* Let $\mathbf{X} = \mathbf{GL}_n \backslash \mathbf{SO}_{2n+1}$, where n is an even number. Here we have $\check{G}_X = \mathrm{Sp}_n \times \mathrm{Sp}_n \subset \mathrm{Sp}_{2n} = \check{G}$ and the valuations corresponding to colors are all simple short roots of $\check{G}$ (with $\sigma_{\check{\theta}} = +, r_{\check{\theta}} = 1$), as well as half the long root of $\check{G}$ (with $\sigma_{\check{\theta}} = +, r_{\check{\theta}} = \frac{1}{2}$). Therefore, L_X (as a conjugation-invariant function on $\check{G}_X$) is up to zeta factors equal to:

$$\frac{L(\pi_1, \frac{1}{2}) L(\pi_1 \otimes \pi_2, 1)}{L(\pi_1, \mathrm{Ad}, 0) L(\pi_2, \mathrm{Ad}, 0)}.$$

7.3. **An analytic criterion.** Notice that, since the image of $S_{\tilde{\chi}^{-1}\nu^{-1}}$ lies in the space of a representation induced from a character of $P(X)$, the morphism $\Delta_{\tilde{\chi}}$ (its adjoint) factors through the quotient: $C_c^\infty(U\backslash G) \to C_c^\infty([P(X), P(X)]\backslash G)$. In fact, a completely analogous theory of Schwartz space and Fourier transforms exists for the space $[P, P]\backslash G$, for any parabolic subgroup P (cf. [BK02]), and therefore $\Delta_{\tilde{\chi}}$ can be viewed as a rational family of morphisms:

$$\mathcal{S}([P(X), P(X)]\backslash G) \to C^\infty(X).$$

Set $\mathbf{P} = \mathbf{P}(\mathbf{X})$, $\mathbf{M} = \mathbf{P}/[\mathbf{P}, \mathbf{P}]$. As in Lemma 3.4.2, $\Delta_{\delta_{(X)}^{\frac{1}{2}} \tilde{\chi}} : \mathcal{S}([P, P]\backslash G) \to C^\infty(X)$ is defined by an integral of the form:

$$\mathrm{ev}_{H1} \circ \Delta_{\tilde{\chi}}(\Phi) = \int_{[P,P]\backslash G} \Phi(x) f_{\tilde{\chi}^{-1}}(x) dx$$

where $f_{\tilde{\chi}^{-1}}$ is a (pseudo-rational) eigenfunction for $M \times H$ with eigencharacter $\chi^{-1} \times 1$ with respect to the normalized M-action (defined by $(L_m \Phi)(x) = \delta_P^{-\frac{1}{2}}(m) \Phi(m \cdot x)$). Since the elements of $\mathcal{S}([P, P]\backslash G)$ are L^2, supported in a compact subset of $\overline{[P, P]\backslash G}^{\mathrm{aff}}$, and hence also L^1, it is clear that this integral converges if $f_{\tilde{\chi}^{-1}}$ is locally bounded. This condition can be expressed in terms of the "valuations" of the pseudo-rational function $f_{\tilde{\chi}^{-1}} \circ \iota$ (where ι denotes the map $g \mapsto g^{-1}$ on $\mathbf{G}$) on the colors of $\mathbf{X}$:

7.3.1. **Lemma.** *The function $f_{\tilde{\chi}^{-1}}$ is locally bounded on $\overline{[P, P]\backslash G}^{\mathrm{aff}}$ if and only if $\left\langle \check{v}_D, \Re(\chi) - \rho_{P(X)} \right\rangle \ge 0$ for all colors $\mathbf{D}$ of $\mathbf{X}$.*

Proof. The absolute value of $f_{\tilde{\chi}^{-1}}$ can be expressed as a product of factors of the form $|g_i|^{s_i}$ where $s_i \in \mathbb{R}$ and $g_i \in k([\mathbf{P}, \mathbf{P}]\backslash \mathbf{G})^{(\mathbf{M}) \times \mathbf{H}} = k(\mathbf{X})^{(\mathbf{P})} \circ \iota$ (where ι denotes the map $g \mapsto g^{-1}$ in $\mathbf{G}$). Such a function $|g_i|$ is bounded if and only if g_i is regular, which happens if and only if $\langle \check{v}_D, -\eta_i \rangle \ge 0$ for all colors $\mathbf{D}$ of $\mathbf{X}$, where η_i is the character of g_i (in additive notation). More generally, the product of $|g_i|^{s_i}$ is bounded if and only if $\sum_i s_i \langle \check{v}_D, -\eta_i \rangle \ge 0$. But the latter is, by definition, equal to $\left\langle \check{v}_D, \Re(\chi) - \rho_{P(X)} \right\rangle$. (Recall that if χ^{-1} is the eigencharacter of $f_{\tilde{\chi}^{-1}}$ with respect

then $\check{v}_D + \check{v}_{D'} = \check{\beta}$. By these rules the calculation of one $\check{v}_D$ implies many others, in many cases (such as the present one) all of them.

to the normalized action of M, its eigenvalue with respect to the unnormalized action is $\delta_{P(X)}^{\frac{1}{2}}\chi^{-1}$.) □

7.3.2. **Corollary.** *The integral expression for $\Delta_{\delta_{(X)}^{\frac{1}{2}}\tilde{\chi}}$ converges if $\Re(\chi)$ lies in $\rho_{P(X)} + \mathcal{T}^\vee$, where $\mathcal{T}$ denotes the cone spanned by all valuations $\check{v}_D$ induced by colors of $\mathbf{X}$ and $\mathcal{T}^\vee \subset \mathcal{X}(\mathbf{X}) \otimes \mathbb{R}$ denotes its dual cone.*

We will now see that extending this result to a larger domain, the *interior* $(\mathcal{T}^\vee)^\circ$ of the whole cone $\mathcal{T}^\vee$, would imply Statement 7.1.2.

7.3.3. **Proposition.** *If all spherical roots γ of $\mathbf{X}$ satisfy Statement 6.2.1, and the integral expression for $\Delta_{\delta_{(X)}^{\frac{1}{2}}\tilde{\chi}} : \mathcal{S}([P(X), P(X)]\backslash G) \to C^\infty(X)$ converges for all $\chi \in (\mathcal{T}^\vee)^\circ$ then Statement 7.1.2 is true for $\mathbf{X}$.*

Proof. Since the B_w are cocycles: $W_X \to \mathbb{C}(A_X^*)$, by Hilbert's Theorem 90 there exists a $\beta \in \mathbb{C}(A_X^*)$ such that $B_w(\tilde{\chi}) = \frac{\beta(\tilde{\chi})}{\beta({}^w\tilde{\chi})}$. This β is uniquely defined up to a W_X-invariant function, and similarly the divisors of any two such β's differ by a W_X-invariant divisor. Let $\mathcal{B}$ be the unique divisor on A_X^* with the following properties:

- $-\mathcal{B}$ is an effective divisor.
- There is no effective W_X-invariant divisor $\mathcal{D}$ such that $-\mathcal{B} - \mathcal{D}$ is effective.
- There is a W_X-invariant divisor $\mathcal{D}$ such that $\mathcal{B} + \mathcal{D}$ is the divisor of β.

In particular, $\mathcal{B}$ has the following property: $\mathcal{B} - {}^{w^{-1}}\mathcal{B} = [B_w]$ for every $w \in W_X$, where $[B_w]$ denotes the divisor of B_w.

We will say that an effective divisor $\mathcal{D}$ "appears" in $\mathcal{B}$ if $-\mathcal{B}-\mathcal{D}$ is effective. Given the form of $[B_w]$, we deduce that $\mathcal{B}$ is a (negative) linear combination of irreducible divisors of the form $[1 \pm q^{-r}e^{\check{\theta}}]$. Indeed, if any irreducible divisor which is not of this form appears, its whole W_X-orbit will have to appear because otherwise it will be an irreducible component of $\mathcal{B} - {}^{w^{-1}}\mathcal{B}$ for some w. But the whole W_X-orbit appearing contradicts the axioms for $\mathcal{B}$. Notice that for a divisor of the form $[1 \pm q^{-r}e^{\check{\theta}}]$ with $r > 0$ the half-line of $\check{\theta}$ in $\mathrm{Hom}(\mathcal{X}(\mathbf{X}), \mathbb{R})$ is uniquely defined. Thus, we will call such a divisor *positive* if $\langle \check{\theta}, \eta \rangle \geq 0$ for all eigencharacters η of $\mathbf{B}$ on $k[\mathbf{X}]^{(\mathbf{B})}$. We claim:

Only positive divisors appear in $\mathcal{B}$.

The poles of $B_w(\tilde{\chi})$ are the same as those of $b_w(\delta_{(X)}^{\frac{1}{2}}\tilde{\chi})$. From the functional equation: $\Delta_{\delta_{(X)}^{\frac{1}{2}}\tilde{\chi}} \circ \mathcal{F}_{w^{-1}} = b_w(\delta_{(X)}^{\frac{1}{2}}\tilde{\chi})\Delta_{\delta_{(X)}^{\frac{1}{2}}{}^w\tilde{\chi}}$, where $\mathcal{F}_{w^{-1}}$ denotes the w^{-1}-Fourier transform on $\mathcal{S}([P(X), P(X)]\backslash G)$ and the fact that the morphisms $\Delta_{\delta_{(X)}^{\frac{1}{2}}\tilde{\chi}}$ are never identically zero we deduce that a divisor D of B_w must be a pole of $\Delta_{\delta_{(X)}^{\frac{1}{2}}\tilde{\chi}}$. Therefore, it cannot intersect the domain of convergence of $\Delta_{\delta_{(X)}^{\frac{1}{2}}\tilde{\chi}}$, which implies that, under our assumption on the domain of convergence, the polar divisor cannot be negative.

Now, it is clear that for every simple spherical root γ the divisors $[1 - \sigma q^{-r}e^{\check{\theta}}]$, where $(\check{\theta}, \sigma, r)$ ranges over all weighted virtual colors which belong to γ, appear in $\mathcal{B}$, since they are pole of B_{w_γ}. We actually claim:

> For every $(\check\theta, \sigma, r)$ in the W_X-orbit of weighted virtual colors which belong to γ the triple $(\check\theta, \sigma, r)$ or the triple $(-\check\theta, \sigma, r)$ appears in $\mathcal{B}$. Moreover, the multiplicity of its appearance is constant for all elements of the W_X-orbit.

In combination with the previous claim, this completely identifies the divisor $\mathcal{B}$ as the divisor of all positive W_X-translates of divisors of the form $[1-\sigma q^{-r} e^{\check\theta}]$ (with the correct multiplicities) where $(\check\theta, \sigma, r)$ is a weighted virtual color. Moreover, it proves that for every such W_X-translate we have $\check\theta > 0$ or $-\check\theta > 0$.

To prove this claim, assume otherwise. Then there exists a $\gamma \in \Delta_X$ and such a pair of opposite triples $(\pm\check\theta, \sigma, r)$ such that neither of the two triples in the pair appears in $\mathcal{B}$, while one of the triples $(\pm^{w_\gamma}\check\theta, \sigma, r)$ appears. But then either of $[1 - \sigma q^{-r} e^{\pm\check\theta}]$ will be a zero of B_{w_γ}. By the form of B_{w_γ} it follows that then $[1 - \sigma q^{-r} e^{\mp\check\theta}]$ is a pole of B_{w_γ}, and therefore appears in $\mathcal{B}$, a contradiction.

To complete the proof of the proposition, we notice that the set of triples $(\check\theta, \sigma, r) \in \Theta^+$ such that $w_\gamma\check\theta < 0$ defines the polar divisor of B_{w_γ}, and therefore consists only of the weighted virtual colors belonging to γ. This proves the desired property for the set Θ^+. $\square$

Part 3. Applications

8. The Hecke module structure, multiplicity one and good test vectors.

From now on we let $\mathbf{X}$ be a homogeneous, *affine* spherical variety which satisfies the conclusion of Theorem 6.2.2 and Statement 7.1.2. Assume also that $\mathring{X}$ has a *unique* B-orbit. In particular, the strong form of the Cartan decomposition holds: $X/K \simeq \Lambda_X^+$ and $A_X^* \subset A^*$. Let $\mathcal{H}_X$ be the quotient of the "spherical Hecke algebra" $\mathcal{H}(G,K)$ which acts on $C_c^\infty(X)^K$. Then, under the Satake isomorphism, the maximal spectrum of $\mathcal{H}_X$ is equal to the image of A_X^* in A^*/W. It was proven in [Sa08] that $C_c^\infty(X)^K$ is torsion-free over $\mathcal{H}_X$. Here we determine precisely the $\mathcal{H}(G,K)$-structure:

8.0.4. **Theorem.** *There is an isomorphism: $C_c^\infty(X)^K \simeq \mathbb{C}[\delta_{(X)}^{\frac12} A_X^*]^{W_X}$, compatible with the $\mathcal{H}(G,K)$-structure and the Satake isomorphism. Under this isomorphism the element $1_{X(\mathfrak{o})}$ is mapped to the constant 1.*

Remark. This proves the conjecture of [Sa08, §6.3] under the assumptions of the present section.

Proof. Let $P_{\check\lambda}$ be the polynomial on $\delta_{(X)}^{\frac12} A_X^*$ defined as $P_{\check\lambda}(\chi\delta_{(X)}^{\frac12}) = \frac{\Omega_{\chi\delta_{(X)}^{\frac12}}(x_{\check\lambda})}{L_{\check X}^{\frac12}(\chi)}$. (In particular, $P_0 = 1$.) Then the map $1_{x_{\check\lambda}K} \mapsto P_{\check\lambda}$ is $\mathcal{H}(G,K)$-equivariant with respect to the Satake isomorphism $\mathcal{H}(G,K) = \mathbb{C}[A^*]^W$. Since all $P_{\check\lambda}$ are W_X-invariant polynomials and generate $\mathbb{C}[\delta_{(X)}^{\frac12} A_X^*]^{W_X}$, the result follows. $\square$

In particular:

8.0.5. **Corollary.** *If* $\dim \mathrm{Hom}_{\mathcal{H}(G,K)}(C_c^\infty(X)^K, \mathbb{C}_\eta) = 1$ *for a* generic *point* $\eta \in \mathrm{spec}_M \mathcal{H}_X$*, then* $\dim \mathrm{Hom}_{\mathcal{H}(G,K)}(C_c^\infty(X)^K, \mathbb{C}_\eta) = 1$ *for* every $\eta \in \mathrm{spec}_M \mathcal{H}_X$ *and* $\dim \mathrm{Hom}_G(\pi, C^\infty(X)) \le 1$ *for* every *irreducible unramified representation* π.

Besides, the previous theorem has the following corollary:

8.0.6. **Theorem** (Good test vectors)**.** *Assume that the unramified spectrum of X is multiplicity-free (which, by the above corollary, is equivalent to saying that it is multiplicity-free for a generic representation in the spectrum). Let π be an irreducible unramified representation and $L : \pi \to \mathbb{C}$ a non-zero H-invariant functional. Then $L|_{\pi^K} \neq 0$.*

Proof. We denote by $\Lambda : \pi \to C^\infty(X)$ the morphism defined from L by Frobenius reciprocity. Let $v \in \pi^K$. Since v generates π and L is non-zero, $\Lambda(v) \neq 0$. In other words, $\left\langle 1_{x_{\check\lambda} K}, \Lambda(v) \right\rangle \neq 0$ for some $\check\lambda \in \Lambda_X^+$. There is $h \in \mathcal{H}(G,K)$ (whose image in $\mathcal{H}_X$ corresponds to the polynomial $P_{\check\lambda}$) such that $1_{x_{\check\lambda}K} = h * 1_{x_0 K}$. Then:

$$\left\langle 1_{x_{\check\lambda}K}, \Lambda(v) \right\rangle = \langle 1_{x_0K}, \Lambda(\pi(h^*)v) \rangle = \eta_\pi(h^*) \langle 1_{x_0K}, \Lambda(v) \rangle = \eta_\pi(h^*) L(v)$$

where η_π is the character of $\mathcal{H}(G,K)$ corresponding to π. Therefore $L(v) \neq 0$. $\square$

8.0.7. **Corollary.** *Let $\mathbf{H} \subset \mathbf{G}$ be reductive groups defined over a global field F, with $\mathbf{H}$ a spherical subgroup of $\mathbf{G}$. Assume that at almost every place v the corresponding group $\mathbf{G}_v$ is split and the variety $\mathbf{H}_v \backslash \mathbf{G}_v$ satisfies the hypotheses of Theorem 8.0.6. Denote by $\mathbb{A}_F$ the ring of adeles of F, and let $\pi \simeq \otimes_v \pi_v$ be a representation of $\mathbf{G}(\mathbb{A}_F)$. If $\mathrm{Hom}_{H_v}(\pi_v, \mathbb{C}) \neq 0$ for every v, then $\mathrm{Hom}_{\mathbf{H}(\mathbb{A}_F)}(\pi, \mathbb{C}) \neq 0$.*

9. Unramified Plancherel formula

We continue to make the same assumptions on $\mathbf{X}$ as in the previous section. Since $\mathbf{H}$ is reductive, we may and will assume that the $\mathbf{G}$-eigenform ω_X on $\mathbf{X}$ is $\mathbf{G}$-invariant, and hence defines a G-invariant measure $|\omega_X|$ on X. Remember that we have normalized that measure so that $|\omega_X|(x_0 J) = 1$. We will compute the Plancherel formula for unramified functions on X. More precisely, it is known that there exists a(n essentially unique) decomposition of $L^2(X)$ as a direct integral of irreducible, unitary representations. Specializing to K-invariant elements of $L^2(X)$, we get for every $\Phi \in C_c^\infty(X)^K$ a formula for $\|\Phi\|^2_{L^2(X)}$ as an integral of $|\langle \Phi, \Omega \rangle|^2$ where Ω ranges over $\mathcal{H}(G,K)$-eigenfunctions belonging to unitary representations, for every $\Phi \in C_c^\infty(X)^K$. We will compute this formula. As an application, we will compute the volume of $\mathbf{X}(\mathfrak{o})$ (essentially, its "Tamagawa volume").

Since we are keeping the assumptions of the previous section, every Hecke eigenfunction is a multiple of

$$\Omega'_\chi := \frac{\Omega_\chi}{L_X^{\frac12}(\chi)}$$

for some $\chi \in \delta_{(X)}^{\frac12} A_X^*$. Notice that $\Omega'_\chi(x_0) = 1$ and $\Omega'_{{}^w\chi} = \Omega'_\chi$ for every $w \in W_X$. Therefore, the Plancherel formula will have the form:

$$\|\Phi\|^2 = \int_{\delta_{(X)}^{\frac12} A_X^*/W_X} |\left\langle \Phi, \Omega'_\chi \right\rangle|^2 d\mu(\chi) \tag{9.1}$$

for every $\Phi \in C_c^\infty(X)^K$ and a *unique* positive measure on $\delta_{(X)}^{\frac12} A_X^*/W_X$ (supported, of course, on the set of points belonging to unitary representations) which, for brevity, when normalized this way, will be called *the* Plancherel measure for X.

9.0.1. **Theorem.** *The Plancherel measure for the unramified spectrum of X is supported on $\delta_{(X)}^{\frac{1}{2}} A_X^{*,1}/W_X$, where $A_X^{*,1}$ denotes the maximal compact subgroup of A_X^*. For every $\Phi \in C_c^\infty(X)^K$ we have:*

$$\|\Phi\|^2 = \frac{1}{Q \cdot |W_X|} \int_{A_X^{*,1}} \left| \left\langle \Phi, \Omega'_{\delta_{(X)}^{\frac{1}{2}}\chi} \right\rangle \right|^2 L_X(\chi) d\chi, \tag{9.2}$$

where $d\chi$ is probability Haar measure on $A_X^{*,1}$ and

$$Q = \frac{\mathrm{Vol}(K)}{\mathrm{Vol}(J w_l J)} = \prod_{\check\alpha \in \check\Phi^+} \frac{1 - q^{-1-\langle \check\alpha, \rho\rangle}}{1 - q^{-\langle \check\alpha, \rho \rangle}}.$$

Remark. With respect to our original eigenfunctions Ω_χ we have:

$$\|\Phi\|^2 = \frac{1}{Q \cdot |W_X|} \int_{A_X^{*,1}} \left| \left\langle \Phi, \Omega_{\delta_{(X)}^{\frac{1}{2}}\chi} \right\rangle \right|^2 d\chi. \tag{9.3}$$

Proof. As in the previous section, let $P_{\check\lambda}$ be the polynomial on $\delta_{(X)}^{\frac{1}{2}} A_X^*$ defined as $P_{\check\lambda}(\chi) = \Omega'_\chi(x_{\check\lambda})$. The fact that $\left\langle 1_{x_0 K}, 1_{x_{\check\lambda} K}\right\rangle_{L^2(X)} = 0$ for $\check\lambda \ne 0$ implies, via the abstract Plancherel formula (9.1), that

$$\int_{\delta_{(X)}^{\frac{1}{2}} A_X^*/W_X} P_{\check\lambda}(\chi) d\mu(\chi) = 0$$

for every $\check\lambda \ne 0$.

Recall that the $P_{\check\lambda}$ span the space $\mathcal{P}$ of polynomials on $\delta_{(X)}^{\frac{1}{2}} A_X^*/W_X$. Given a linear functional on this space of polynomials, there exists at most one real-valued measure of bounded support on $\delta_{(X)}^{\frac{1}{2}} A_X^*/W_X$ which represents this functional. (Indeed, this follows from the density of the functions of the form $\Re P, \Im P$, $P \in \mathcal{P}$, in the space of continuous functions on any compact domain.) We will show in a combinatorial way that $[1_{x_{\check\lambda} K}, 1_{x_0 K}] = 0$ for $\check\lambda \ne 0$, where $[,]$ denotes the hermitian inner product defined by the right hand side of (9.2) (or (9.3)). It will then follow that Plancherel measure is a multiple of the measure of (9.2).

9.0.2. **Lemma.** *For every $\check\lambda \ne 0$ we have: $[1_{x_{\check\lambda} K}, 1_{x_0 K}] = 0$, where $[,]$ denotes the hermitian inner product defined by the right hand side of (9.2).*

It suffices to show that $\int_{A_X^{*,1}} P_{\check\lambda}(\delta_{(X)}^{\frac{1}{2}}\chi) L_X(\chi) d\chi = 0$ for $\check\lambda \ne 0$. The value of this integral is equal to the constant term of the Laurent series expansion of the rational function $P_{\check\lambda}(\delta_{(X)}^{\frac{1}{2}}\chi) L_X(\chi)$ (see, for instance, [Ma01]). We have:

$$P_{\check\lambda}(\delta_{(X)}^{\frac{1}{2}}\chi) L_X(\chi) = c \frac{\prod_{\check\gamma \in \check\Phi_X} (1 - e^{\check\gamma})}{\prod_{\check\theta \in \Theta} (1 - \sigma_{\check\theta} q^{-r_{\check\theta}} e^{\check\theta})} \sum_{W_X} \frac{\prod_{\check\theta \in \Theta^+} (1 - \sigma_{\check\theta} q^{-r_{\check\theta}} e^{\check\theta})}{\prod_{\check\gamma \in \check\Phi_X^+} (1 - e^{\check\gamma})} e^{\check\lambda} \left({}^w\chi\right) =$$

$$= c \sum_{W_X} \frac{\prod_{\check\gamma \in \check\Phi_X^-} (1 - e^{\check\gamma})}{\prod_{\check\theta \in \Theta^-} (1 - \sigma_{\check\theta} q^{-r_{\check\theta}} e^{\check\theta})} e^{\check\lambda} \left({}^w\chi\right).$$

In the notation of the proof of Theorem 7.2.1, all weights in the Laurent expansion of $\frac{\prod_{\check\gamma\in\check\Phi_X^-}(1-e^{\check\gamma})}{\prod_{\check\theta\in\Theta^-}(1-\sigma_{\check\theta}q^{-r_{\check\theta}}e^{\check\theta})}e^{\check\lambda}$ are $\prec_2 \check\lambda$. Therefore, 0 is a weight of the above expression only if $0\in w(\check\lambda-\mathcal{T})$ for some $w\in W_X$, equivalently only if $\check\lambda\in\mathcal{T} \iff \check\lambda=0$. This proves the lemma.

Hence the Plancherel formula has to be a multiple of the right-hand-side of (9.2). On the other hand, a method of Bernstein (discussed in [SV]) implies that the right-hand-side of (9.2) is precisely the *most continuous part* of the Plancherel formula. The theorem follows. ☐

The Plancherel formula has the following corollary.[12]

9.0.3. **Theorem.** *The measure of* $\mathbf{X}(\mathfrak{o})$ *is:*

$$|\omega_X|(\mathbf{X}(\mathfrak{o})) = Q\cdot c^{-1} = Q\frac{\prod_{\check\gamma\in\check\Phi_X^+}(1-e^{\check\gamma})}{\prod_{\check\theta\in\Theta^+}(1-\sigma_{\check\theta}q^{-r_{\check\theta}}e^{\check\theta})}(\delta_{P(X)}^{\frac12}). \tag{9.4}$$

Proof. We have:

$$|\omega_X|(\mathbf{X}(\mathfrak{o})) = \|1_{\mathbf{X}(\mathfrak{o})}\|^2 = \frac{1}{Q\cdot|W_X|}\int_{A_X^{*,1}}\left|\left\langle 1_{\mathbf{X}(\mathfrak{o})},\Omega'_{\delta_{(X)}^{\frac12}\chi}\right\rangle\right|^2 L_X(\chi)d\chi =$$

$$= \frac{1}{Q\cdot|W_X|}\int_{A_X^{*,1}}|P_0(\delta_{(X)}^{\frac12}\chi)|^2\left(|\omega_X|(\mathbf{X}(\mathfrak{o}))\right)^2 L_X(\chi)d\chi \Rightarrow$$

$$|\omega_X|(\mathbf{X}(\mathfrak{o})) = Q\cdot|W_X|\left(\int_{A_X^{*,1}}L_X(\chi)d\chi.\right)^{-1}$$

As before, the integral of $L_X(\chi)=P_0(\delta_{(X)}^{\frac12}\chi)L_X(\chi)$ is equal to the constant term in the Laurent expansion of:

$$c\sum_{W_X}\frac{\prod_{\check\gamma\in\check\Phi_X^-}(1-e^{\check\gamma})}{\prod_{\check\theta\in\Theta^-}(1-\sigma_{\check\theta}q^{-r_{\check\theta}}e^{\check\theta})}$$

which in this case is equal to $c\cdot|W_X|$. Therefore:

$$|\omega_X|(\mathbf{X}(\mathfrak{o})) = Q\cdot c^{-1} = Q\cdot\frac{\prod_{\check\gamma\in\check\Phi_X^+}(1-e^{\check\gamma})}{\prod_{\check\theta\in\Theta^+}(1-\sigma_{\check\theta}q^{-r_{\check\theta}}e^{\check\theta})}(\delta^{\frac12}).$$ ☐

Remark. It is natural to call the measure $(1-q^{-1})^{\mathrm{rk}(A_X)}\cdot|\omega_X|$ the *Tamagawa measure* on X. Indeed, any globally defined invariant volume form on $\mathbf{X}$ will induce, for almost every place, the same measure on $\mathring{X}$ as $\mathfrak{q}^*(a^{-1}da)\wedge du$ (where $\mathfrak{q}:\mathring{\mathbf{X}}\to\mathbf{A}_X$ is the natural projection based at the point x_0). With respect to the latter, we have $\mathrm{Vol}(x_0J=x_0B_0)=a^{-1}da(A_{X0})=(1-q^{-1})^{\mathrm{rk}(A_X)}$, therefore the "Tamagawa measure" $d^{\mathrm{Tam}}x$ is equal to $(1-q^{-1})^{\mathrm{rk}(A_X)}|\omega_X|$.

Therefore we have shown:

$$d^{\mathrm{Tam}}x(\mathbf{X}(\mathfrak{o})) = Q\cdot(1-q^{-1})^{\mathrm{rk}(A_X)}\cdot\frac{\prod_{\check\gamma\in\check\Phi_X^+}(1-e^{\check\gamma})}{\prod_{\check\theta\in\Theta^+}(1-\sigma_{\check\theta}q^{-r_{\check\theta}}e^{\check\theta})}(\delta^{\frac12}). \tag{9.5}$$

[12] While everything up to this point can be rephrased for the case of non-trivial line bundles $\mathcal{L}_\Psi$, clearly Theorem 9.0.3 does not make sense in that case.

10. Periods of Eisenstein series.

Let now F be a number field and $\mathbf{G}$ a split connected reductive group defined over the ring of integers of F. We use the same notation: $\mathbf{B}$, $\mathbf{A}$, etc. as above and assume that $\mathbf{B}(F)$ has a single orbit on $\mathring{\mathbf{X}}(F)$. We also assume that for almost all completions F_v of F the variety $\mathbf{X}_{F_v}$ satisfies the assumptions of the previous two sections. We will denote by $\mathbf{A}(\mathbb{A}_F)^1$ (and similarly for other groups than $\mathbf{A}$) the intersection of the kernels of all homomorphisms: $\mathbf{A}(\mathbb{A}_F) \to \mathbb{G}_m(\mathbb{A}_F) \to \mathbb{R}_+^\times$ where the first arrow denotes an algebraic character and the second denotes the absolute value. Hence, $\mathbf{A}(\mathbb{A}_F)/\mathbf{A}(\mathbb{A}_F)^1 \simeq (\mathbb{R}_+^\times)^{\mathrm{rk}(\mathbf{A})}$. For every place v we denote the group $\mathbf{G}(F_v)$ by G_v, the group $\mathbf{G}(\mathfrak{o}_v)$ (if $v < \infty$) by K_v, etc.

Any idele class character of $\mathbf{A}(\mathbb{A}_F)$ can be twisted by characters of the group $\mathbf{A}(\mathbb{A}_F)/\mathbf{A}(\mathbb{A}_F)^1$, and thus lives in a $\mathrm{rk}(\mathbf{A})$-dimensional complex manifold of characters. Let ω denote such a family. For $\chi \in \omega$ we denote by $I(\chi) = \mathrm{Ind}_{\mathbf{B}(\mathbb{A}_F)}^{\mathbf{G}(\mathbb{A}_F)}(\chi\delta^{\frac{1}{2}})$ the normalized principal series[13] of $\mathbf{G}(\mathbb{A}_F)$, considered as a holomorphic family of vector spaces. For a meromorphic family of sections $\chi \mapsto f_\chi \in I(\chi)$ we have the principal Eisenstein series defined by the convergent sum:

$$E(f_\chi, g) = \sum_{\gamma \in \mathbf{B}(F)\backslash \mathbf{G}(F)} f_\chi(\gamma g) \tag{10.1}$$

if $\langle \check{\alpha}, \Re(\chi)\rangle \gg 0$ for all $\alpha \in \Delta$, and by meromorphic continuation to the whole ω.

Let $\mathbf{H}$ be a spherical subgroup of $\mathbf{G}$ over F. We would like to compute the period integral $\int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A}_F)} E(f_\chi, h)dh$. Of course, this integral may not be convergent, therefore we have to make sense of it as a distribution on the Eisenstein spectrum of $\mathbf{G}$. Our goal is to compute the most continuous part of this distribution. For simplicity of notation, we will discuss only the case where ω consists of the characters of $\mathbf{A}(\mathbb{A}_F)/\mathbf{A}(\mathbb{A}_F)^1$; however, the argument and the result hold for any family of idele class characters, with the only modification that the summation in (10.5) is only over orbits of maximal rank for which ω is trivial on $\mathbf{B}_\xi(\mathbb{A}_F)^1$ (in the notation that follows).

In what follows, we use Tamagawa measures for all groups. It is convenient to do formal computations with measures given by differential forms, without convergence factors, and interpret the non-convergent formal products in the end as special valued of zeta functions or residues thereof (if $\zeta(1)$ appears, where ζ denotes the Dedekind zeta function of F). Let $\Phi \in \text{c-Ind}_{\mathbf{A}(\mathbb{A}_F)^1\mathbf{U}(\mathbb{A}_F)}^{\mathbf{G}(\mathbb{A}_F)}(1)$ (where c-Ind denotes compact induction), and write Φ in terms of its Mellin transform with respect to the left-$\mathbf{A}(\mathbb{A}_F)$-action:

$$\Phi(g) = \int_{\widehat{\mathbf{A}(\mathbb{A}_F)/\mathbf{A}(\mathbb{A}_F)^1}} f_{\chi\delta^{-\frac{1}{2}}}(g) d\chi$$

where $f_\chi \in I(\chi)$ and $d\chi$ is Haar measure on the unitary dual of $\mathbf{A}(\mathbb{A}_F)/\mathbf{A}(\mathbb{A}_F)^1$. Note that the unitary dual of $\mathbf{A}(\mathbb{A}_F)/\mathbf{A}(\mathbb{A}_F)^1$ can naturally be identified with the imaginary points $i\mathfrak{a}_\mathbb{R}^*$ of the Lie algebra of the dual torus, via an isomorphism which we will denote by exp, i.e. $\widehat{\mathbf{A}(\mathbb{A}_F)/\mathbf{A}(\mathbb{A}_F)^1} = \exp(i\mathfrak{a}_\mathbb{R}^*)$. Notice also the shift $\delta^{-\frac{1}{2}}$ in the above formula because f_χ has been defined with respect to the *normalized* action of $\mathbf{A}(\mathbb{A}_F)$. However, by our assumption that Φ is compactly supported

[13] Our convention for the archimedean places will be that Ind denotes the K_∞-finite vectors in that representation, where K_∞ is a maximal compact subgroup of G_∞.

modulo $\mathbf{A}(\mathbb{A}_F)^1\mathbf{U}(\mathbb{A}_F)$, its Mellin transform is entire in χ and hence we can shift the contour of integration and write:

$$\Phi(g) = \int_{\exp(\kappa+i\mathfrak{a}^*_{\mathbb{R}})} f_\chi(g)d\chi$$

for any $\kappa \in \mathfrak{a}^*_{\mathbb{C}}$. In particular, we can shift to the domain of convergence of the Eisenstein sum (10.1) and then we will have:

$$\sum_{\gamma\in\mathbf{B}(F)\backslash\mathbf{G}(F)} \Phi(\gamma g) = \int_{\exp(\kappa+i\mathfrak{a}^*_{\mathbb{R}})} E(f_\chi, g)d\chi,$$

a function of rapid decay on the automorphic quotient $\mathbf{G}(F)\backslash\mathbf{G}(\mathbb{A}_F)$ (cf. [MW94, II.1.11]).

We integrate over $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A}_F)$:

$$\int_{\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A}_F)} \sum_{\gamma\in\mathbf{B}(F)\backslash\mathbf{G}(F)} \Phi(\gamma h)dh =$$

$$= \sum_{\xi\in[\mathbf{B}(F)\backslash\mathbf{G}(F)/\mathbf{H}(F)]} \int_{\mathbf{H}_\xi(\mathbb{A}_F)\backslash\mathbf{H}(\mathbb{A}_F)} \int_{\mathbf{H}_\xi(F)\backslash\mathbf{H}_\xi(\mathbb{A}_F)} \int_{\exp(\kappa+i\mathfrak{a}^*_{\mathbb{R}})} f_\chi(\xi a h)d\chi da dh. \tag{10.2}$$

Here $[\mathbf{B}(F)\backslash\mathbf{G}(F)/\mathbf{H}(F)]$ denotes a (finite) set of representatives in $\mathbf{G}(F)$ for the $(\mathbf{B}(F), \mathbf{H}(F))$-double cosets, and $\mathbf{H}_\xi := \mathbf{H} \cap \xi^{-1}\mathbf{B}\xi$. Similarly we will denote: $\mathbf{B}_\xi := \mathbf{B} \cap \xi\mathbf{H}\xi^{-1}$, and we will let $\mathbf{Y}$ denote the $\mathbf{B}$-orbit of $\xi\mathbf{H}$ on $\mathbf{G}/\mathbf{H}$. Here the measure da is a (Tamagawa) $\mathbf{H}_\xi(\mathbb{A}_F)$-eigenmeasure, with eigencharacter the inverse of the modular character of $\mathbf{H}_\xi$, so that the integral over $\mathbf{H}$ admits a factorization as above. We will denote this character by η_ξ, and its ξ-conjugate – which is a character of $\mathbf{B}_\xi(\mathbb{A}_F)$ – by η_Y. Hence the two inner integrals are valued in the line bundle over $\mathbf{H}_\xi(\mathbb{A}_F)\backslash\mathbf{H}(\mathbb{A}_F)$ defined by η_ξ^{-1}, and dh is an $\mathbf{H}(\mathbb{A}_F)$-invariant measure valued in the dual of that line bundle.

For a fixed $h \in \mathbf{H}(\mathbb{A}_F)$ the two inner integrals:

$$\int_{\mathbf{H}_\xi(F)\backslash\mathbf{H}_\xi(\mathbb{A}_F)} \int_{\exp(\kappa+i\mathfrak{a}^*_{\mathbb{R}})} f_\chi(\xi a h)d\chi da = \int_{\mathbf{B}_\xi(F)\backslash\mathbf{B}_\xi(\mathbb{A}_F)} \int_{\exp(\kappa+i\mathfrak{a}^*_{\mathbb{R}})} f_\chi(a\xi h)d\chi da$$

are equal to:

$$\mathrm{Vol}(\mathbf{B}_\xi(F)\backslash\mathbf{B}_\xi(\mathbb{A}_F)^1) \cdot \int_{\mathbf{B}_\xi(\mathbb{A}_F)^1\backslash\mathbf{B}_\xi(\mathbb{A}_F)} \int_{\exp(\kappa+i\mathfrak{a}^*_{\mathbb{R}})} f_\chi(a\xi h)d\chi da.$$

The Tamagawa volume that appears here is equal to 1, since $\mathbf{B}_\xi$ is a connected, split, solvable group. By abelian Fourier analysis the last expression is equal to:

$$\int_{\delta^{-\frac{1}{2}}\eta_Y^{-1}\exp(i\mathfrak{a}^*_{Y,\mathbb{R}})} f_\chi(\xi h)d\chi,$$

where we have taken into account that $\exp(\mathfrak{a}^*_Y)$, where $\mathfrak{a}^*_Y$ is the Lie algebra of A^*_Y, is the orthogonal complement of $\mathbf{B}_\xi(\mathbb{A}_F)$ in $\exp(\mathfrak{a}^*)$.

To determine the most continuous part of the $\mathbf{H}$-period integral, hence, it is enough to consider those ξ which correspond to orbits $\mathbf{Y}$ of maximal rank. Again, we can move the contour of integration, this time to $\exp(\kappa_Y + i\mathfrak{a}^*_{Y,\mathbb{R}})$, where κ_Y

is deep in the region where the morphisms $\Delta^Y_{\chi,v}$ introduced in the present paper[14] (but now with an index v to indicate the place of F) are convergent. Returning to (10.2), we can interchange the order of integration to express the contribution of the orbit $\mathbf{Y}$ as:

$$\int_{\exp(\kappa_Y+i\mathfrak{a}^*_{Y,\mathbb{R}})}\int_{\mathbf{H}_\xi(\mathbb{A}_F)\backslash\mathbf{H}(\mathbb{A}_F)} f_\chi(\xi h)dhd\chi$$

and the new inner integral is equal to $\prod_v \Delta^{Y,\mathrm{Tam}}_{\chi,v}$ (where the exponent $^{\mathrm{Tam}}$ stands to show that we are using Tamagawa measures, rather than the measures used throughout the paper). Interchanging the order of summation is justified as follows: The function

$$\Phi'(a) := \exp(\kappa_Y)(a)\int_{\mathbf{H}_\xi(\mathbb{A}_F)^1\backslash\mathbf{H}(\mathbb{A}_F)}\Phi(a\xi h)dh,$$

on $\mathbf{A}_Y(\mathbb{A}_F)$ is a Schwartz–Harish-Chandra function if κ_Y is sufficiently deep in the domain of convergence of the morphisms $\Delta^Y_{\chi,v}$. Therefore, its value at 1 (which is represented by the iterated integral above before changing the order of integration) is equal to the integral of its Mellin transforms. But those are given by the double integral of $\Phi \in C_c^\infty(\mathbf{U}(\mathbb{A}_F)\mathbf{A}(\mathbb{A}_F)^1\backslash\mathbf{G}(\mathbb{A}_F))$ over the corresponding orbit of the group $\mathbf{A}(\mathbb{A}_F)/\mathbf{A}(\mathbb{A}_F)^1 \times \mathbf{H}(\mathbb{A}_F)$ and against a character of that subgroup; for characters of the form $\exp(\kappa_Y + i\mathfrak{a}^*_{Y,\mathbb{R}})$ similar considerations as in the local case imply that this double integral is absolutely convergent, and therefore equal to:

$$\int_{\mathbf{H}_\xi(\mathbb{A}_F)\backslash\mathbf{H}(\mathbb{A}_F)} f_\chi(\xi h)dh.$$

Fix a finite set of places S, including the infinite ones and those finite places where our assumptions on the spherical variety $\mathbf{X}=\mathbf{H}\backslash\mathbf{G}$ do not hold, such that we have a factorization: $f_\chi=\prod_v f_{\chi,v}$ with $f_{\chi,v}$ being the "standard" K_v-invariant function: $f^0_{\chi,v}(bk)=\chi\delta^{\frac12}(b)$ (denoted by $\phi_{K,\chi}$ in §4.3) for every $v\notin S$. For each orbit $\mathbf{Y}$ of maximal rank, choose an element $w\in W$ with $\mathbf{Y}={}^w\mathring{\mathbf{X}}$ and the length of w equal to the codimension of $\mathbf{Y}$ (in the notation of §3.3: $w\in W(\mathbf{Y})^{-1}$). We can then write $f^0_{\chi,v}$ as $j^{-1}_{w,v}({}^{w^{-1}}\chi)T_w f^0_{{}^{w^{-1}}\chi,v}$, where $j_{w,v}(\chi):=\prod_{\check\alpha>0,w\check\alpha<0}\frac{1-q^{-1}e^{\check\alpha}(\chi_v)}{1-e^{\check\alpha}(\chi_v)}$ at the finite place with residual degree q (we are using the standard intertwining operators T_w here, cf. (4.7)). We will also set $\tilde{j}_{w,v}(\chi):=\prod_{\check\alpha>0,w\check\alpha>0}\frac{1-q^{-1}e^{\check\alpha}(\chi_v)}{1-e^{\check\alpha}(\chi_v)}$ Using the fact that $\Delta^{Y,\mathrm{Tam}}_{\chi,v}\circ T_w=\Delta^{\mathrm{Tam}}_{{}^{w^{-1}}\chi,v}$ to express the contribution of the orbit $\mathbf{Y}$ to (10.2) as:

$$\int_{\exp(\kappa_Y+i\mathfrak{a}^*_{Y,\mathbb{R}})} j^S_w({}^{w^{-1}}\chi)^{-1}\prod_{v\notin S}\Delta^{\mathrm{Tam}}_{{}^{w^{-1}}\chi,v}(f^0_{{}^{w^{-1}}\chi,v})\prod_{v\in S}\Delta^{Y,\mathrm{Tam}}_{\chi,v}(f_{\chi,v})d\chi.$$

Notice that the domain of convergence of $\Delta_{\chi,v}$ contains a translate of the cone of regular eigenfunctions, which is the cone $\mathcal{T}^\vee\subset\mathcal{X}(\mathbf{X})\otimes\mathbb{R}$ dual to the cone $\mathcal{T}$ spanned by the valuations $\check v_D$ of all colors. Since the cone of colors contains the images of the positive co-roots of $\mathbf{G}$, we have simultaneous convergence for $\Delta_{\chi,v}$ and T_w (acting on $I(\chi_v)$). It is easy to argue, based on the fact that the map

[14] Though we have not discussed archimedean places here, all definitions and properties of meromorphic – not rational now – continuation can easily be established in a similar manner. Notice also that the convergence, for $\Re(\chi)$ in a certain cone, of the product over all places v of the operators $\Delta^Y_{\chi,v}$, considered as an $\mathbf{H}(\mathbb{A}_F)$-invariant functional on $C_c^\infty(\mathbf{U}\backslash\mathbf{G}(\mathbb{A}_F))$, is established by the same argument as in the local case.

$\mathbf{B}w^{-1}\mathbf{B} \times^{\mathbf{B}} \mathbf{Y} \to \mathbf{X}$ is birational [Br01, Lemma 5] that whenever $\Delta_{\chi,v}$ and T_w (acting on $I(\chi)$) converge simultaneously, $\Delta^Y_{{}^w\chi,v}$ also converges. Therefore, if we substitute χ by ${}^w\chi$, the domain of integration now can be taken to be $\exp(\kappa+i\mathfrak{a}^*_{X,\mathbb{R}})$, where $\kappa \in \rho_{(X)} + \mathfrak{a}^*_{X,\mathbb{C}}$ is deep in the domain of convergence of the integral for $\Delta_{\chi,v}$. The Eisenstein sum also converges in that region.

We discuss how $\Delta^{\mathrm{Tam}}_{\chi,v}$ and $\Delta_{\chi,v}$ are related:

10.0.1. **Lemma.** *We have:*

$$\Delta^{\mathrm{Tam}}_{\chi,v} = Q_v \Delta_{\chi,v}. \tag{10.3}$$

Proof. As we did in the case of $\Delta_{\chi,v}$, we compose the morphism $\Delta^{\mathrm{Tam}}_{\chi,v}$ with the map $\mathcal{S}(U\backslash G) \to I(\chi)$ (integration against an A-eigenmeasure with $da(A_0) = 1$) to get a morphism: $\mathcal{S}(U\backslash G) \to C^\infty(X)$. Notice that $da = (1-q^{-1})^{-\mathrm{rk}(A)} d^{\mathrm{Tam}}a$. Therefore, the integral expression for $\Delta^{\mathrm{Tam}}_{\chi,v}$ on $U\backslash G$ is an integral, over the open $A\times H$-orbit, against $(1-q^{-1})^{-\mathrm{rk}(A)}$ times a Tamagawa eigenmeasure. On the other hand, the morphisms $\Delta_{\chi,v}$ were defined using an $A \times H$-eigenfunction times the G-invariant measure dx on $U\backslash G$ such that: $dx(U\backslash UK) = 1$. Therefore, if dx^{Tam} denotes "Tamagawa" measure on $U\backslash G$ then we have:

$$\Delta^{\mathrm{Tam}}_{\chi,v} = (1-q^{-1})^{-\mathrm{rk}(A)} \frac{dx^{\mathrm{Tam}}}{dx} \Delta_{\chi,v}.$$

We compute: $dx^{\mathrm{Tam}}(U\backslash Uw_lJ) = dx^{\mathrm{Tam}}(B_0) = (1-q^{-1})^{\mathrm{rk}(A)}$. Therefore:

$$\Delta^{\mathrm{Tam}}_{\chi,v} = \frac{1}{dx(U\backslash Uw_lJ)} \Delta_{\chi,v} = Q_v \Delta_{\chi,v}.$$

□

Therefore:

$$\Delta^{\mathrm{Tam}}_{\chi,v}(f^0_{\chi,v}) = Q_v \Delta_{\chi,v}(f^0_{\chi,v}) = \prod_{\check\alpha>0} \frac{1-q^{-1}e^{\check\alpha}}{1-e^{\check\alpha}}(\chi)\Omega_{\chi,v}(x_0) =$$

$$= \prod_{\check\alpha>0} \frac{1-q^{-1}e^{\check\alpha}}{1-e^{\check\alpha}}(\chi) \cdot L^{\frac12}_{\chi,v}$$

Recall that $L^{\frac12}_{X,v}(\chi) = c_v \cdot \beta_v(\chi)$, where c_v is a quotient of products of local values for the Dedekind zeta function of F and β_v is a quotient of products of Dirichlet L-values which depend on χ. If we consider the product $\prod_{v\notin S} c_v$ it may not converge in general. However, we can make sense of it by considering the leading term of its Laurent expansion, when considered as a specialization of a product/quotient of translates of ζ^S. We will denote this number by $\left(c^S\right)^*$. The standard definition of Tamagawa measures implies that when computing the product over all $v \notin S$ in the expression above, we should use $\left(c^S\right)^*$ wherever c^S formally appears in the product. Similarly, we will denote: $\left(L^{\frac12,S}_X\right)^* = (c^S)^* \cdot \beta_v(\chi)$. Therefore we get:

10.0.2. **Theorem.** *The period integral of:*

$$\sum_{\gamma\in\mathbf{B}(F)\backslash\mathbf{G}(F)} \Phi(\gamma g) = \int_{\exp(\kappa+i\mathfrak{a}^*_{\mathbb{R}})} E(f_\chi, g) d\chi \tag{10.4}$$

over $\mathbf{H}(F)\backslash\mathbf{H}(\mathbb{A}_F)$ is equal to:

$$\int_{\exp(\kappa+i\mathfrak{a}^*_{X,\mathbb{R}})}\left(L_X^{\frac{1}{2},S}(\chi)\right)^*\sum_{[W/W_{(X)}]}\left(\tilde{j}_w^S(\chi)\prod_{v\in S}\Delta^{Y,\mathrm{Tam}}_{^w\chi,v}(f_{^w\chi,v})\right)d\chi \tag{10.5}$$

plus terms which depend on the restriction of f_χ, as a function of χ, to a subvariety of smaller dimension. Here $[W/W_{(X)}]$ denotes a set of representatives of minimal length for $W/W_{(X)}$-cosets, $\kappa\in\rho_{(X)}+\mathfrak{a}^*_{X,\mathbb{C}}$ is deep in the domain of convergence of Δ_χ, and f_χ, the Mellin transform of Φ with respect to the normalized left $\mathbf{A}(\mathbb{A}_F)/\mathbf{A}(\mathbb{A}_F)^1$-action, is assumed to be factorizable with factors $f^0_{\chi,v}$ for $v\notin S$.

Remarks. (1) It appears as if the above expression depends on the choice of representatives $w\in W(\mathbf{Y})^{-1}$. However, the factors $c_w^S(\chi)$, for $\chi\in A_X^*$, do not depend on the choice w. This is easily seen in the case of $\mathbf{PGL}_2\backslash(\mathbf{PGL}_2\times\mathbf{PGL}_2)$, and the general case can be reduced to that by Brion's analysis of Knop's graph (cf. Proposition 2.3.1).

(2) For the special case of the spherical variety $\mathbf{H}\backslash(\mathbf{H}\times\mathbf{H})$, where all $\mathbf{B}$-orbits are of maximal rank and hence the above formula is precise, compare with the calculation of the scalar product of two pseudo-Eisenstein series in [MW94, II.2.1].

(3) One can continue along these lines and give a new argument for the Tamagawa number of $\mathbf{H}$, which would specialize to the argument of [Lan66] in the group case; however, since the setup for this argument is much more general, we will present it in separate work.

What about period integrals of cusp forms? Based on the work of Waldspurger [Wal85], Ichino and Ikeda [II], we formulate in [SV] a conjecture which links the period integral of cusp forms to the local Plancherel formula. More precisely, the "canonical" global functional: "$\mathbf{H}$-period on π times $\mathbf{H}$-period on $\tilde{\pi}$" (where π denotes the space of a cuspidal automorphic representation and $\tilde{\pi}$ its dual) is conjectured to be related to the Plancherel measure on $\mathbf{X}$ and hence, through our computation of §9, to L_X.

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